***Manonmaniam Sundaranar University,***

***Directorate of Distance & Continuing Education,***

***Tirunelveli - 627 012 Tamilnadu, India***

***OPEN AND DISTANCE LEARNING (ODL) PROGRAMMES***

*(FOR THOSE WHO JOINED THE PROGRAMMES FROM THE ACADEMIC YEAR 2023–2024)*

**I YEAR**

# *M.Sc. Physics*

***Course Material***

# *Mathematical Physics*

***Prepared***

***By***

***Dr. V. Sabarinathan***

***Dr. S. Arunavathi***

**Department of Physics**

**Manonmaniam Sundaranar University**

**Tirunelveli - 12**

**M.Sc. PHYSICS – I YEAR**

***SPHM11* : MATHEMATICAL PHYSICS**

**SYLLABUS**

**UNIT I: LINEAR VECTOR SPACE**

Basic concepts – Definitions- examples of vector space – Linear independence - Scalar product- Orthogonality – Gram-Schmidt orthogonalization procedure – linear operators – Dual space- ket and bra notation – orthogonal basis – change of basis – Isomorphism of vector space – projection operator –Eigen values and Eigen functions–Directs man din variant sub space–orthogonal Transformations and rotation

**UNITII: COMPLEX ANALYSIS, PROBABILITY&STATISTICS**

Review of Complex Numbers - de Moivre's Theorem - Functions of a Complex Variable- Differentiability - Analytic functions- Harmonic Functions- Complex Integration- Contour Integration, Cauchy – Riemann conditions – Singular points – Cauchy's Integral Theorem and integral Formula -Taylor's Series - Laurent's Expansion-Zeros and poles – Residue theorem.
Probability – Introduction – Addition rule of probability – Multiplication law of probability– Problems – Introduction to statistics – Mean, median, mode and standard deviations.

**UNIT III: MATRICES**

Types of Matrices and their properties, Rank of a Matrix-Conjugate of a matrix
- Adjoint of a matrix - Inverse of a matrix - Hermitian and Unitary Matrices - Trace of a matrix- Transformation of matrices - Characteristic equation - Eigen values and Eigen vectors - Cayley–Hamilton theorem – Diagonalization

**UNIT IV: FOURIER TRANSFORMS & LAPLACE TRANSFORMS**

Definitions - Fourier transform and its inverse - Transform of Gaussian function and Dirac delta function - Fourier transform of derivatives - Cosine and sine transforms - Convolution theorem. Application: Diffusion equation: Flow of heat in an infinite and in a semi - infinite medium - Wave equation: Vibration of an infinite string and of a semi - infinite string.
Laplace transform and its inverse - Transforms of derivatives and integrals – Differentiation and integration of transforms - Dirac delta functions -Application - Laplace equation: Potential problem in a semi - infinite strip

**UNIT V: DIFFERENTIAL EQUATIONS**

Second order differential equation – Sturm - Liouville's theory - Series solution with simple examples - Hermite polynomials - Generating function - Orthogonality properties - Recurrence relations – Legendre polynomials – Generating function - Rodrigue formula – Orthogonality properties - Dirac delta function – One dimensional Green's function and Reciprocity theorem-Sturm - Liouville's type equation in one dimension & their Green's function.

**UNIT VI: PROFESSIONAL COMPONENTS**

Expert Lecture, Online seminar – Webinars on Industrial Interactions/Visits, Competitive Examinations, Employable and Communication Skill Enhancement, Social Accountability and Patriotism

**UNIT I: LINEAR VECTOR SPACE**

Basic concepts – Definitions- examples of vector space – Linear independence - Scalar product- Orthogonality – Gram-Schmidt orthogonalization procedure – linear operators – Dual space- ket and bra notation – orthogonal basis – change of basis – Isomorphism of vector space – projection operator –Eigen values and Eigen functions– Directs man din variant sub space–orthogonal Transformations and rotation

## 1.1 Vectors:

A vector is a quantity having both magnitude and direction such as force, velocity, acceleration, displacement etc.

## 1.2 Vector space:

Let (F, +) be a field. Let V be a non empty set whose elements are vectors. Then V is a vector space over the field F, if the following conditions are satisfied:

1. (V, +) is an abelian group

(i) Closure property:

V is closed with respect to addition

i.e., $\alpha \in V, \beta \in V \Rightarrow \alpha + \beta \in V$

(ii) Associative:

$\alpha + (\beta + \Upsilon) = (\alpha + \beta) + \Upsilon, \forall \alpha, \beta, \Upsilon \in V$

(iii) Existence of identity:

$\exists$ an elements $0 \in V$ (zero vector) such that $\alpha + 0 = \alpha, \forall \alpha \in V$

(iv) Existence of inverse:

To every vector $\alpha$ in V can be associated with a unique vector $-\alpha$ in V called the additive inverse

i.e., $\alpha + (-\alpha) = 0$

(v) Commutative:

$\alpha + \beta = \beta + \alpha, \forall \alpha, \beta \in V$

2. V is closed under scalar multiplication

i.e., $a \in F, \alpha \in V \Rightarrow a\,\alpha \in V$

3. Multiplication and addition of vector is a distributive property i.e.,

(i) $a(\alpha + \beta) = a\alpha + a\beta, \ \forall\ a \in F, \alpha, \beta \in V$

(ii) $(a + b)\alpha = a\alpha + b\alpha, \ \forall\ a, b \in F, \alpha \in V$

(iii) $(ab)\alpha = a(b\alpha), \ \forall\ a, b \in F, \alpha \in V$

(iv) $1 \cdot \alpha = \alpha, \ \forall\ \alpha \in V$ and 1 is the unity element in F.

## 1.3 Linear dependence and independence of vectors

Vectors (matrices) $X_1, X_2, .... X_n$ are said to be dependent. if

(1) all the vectors (row or column matrices) are of the same order.

(2) $n$ scalars $\lambda_1, \lambda_2, ... \lambda_n$ (not all zero) exist such that

$$\lambda_1 X_1 + \lambda_2 X_2 + \lambda_3 X_3 + ..... + \lambda_n X_n = 0$$

Otherwise they are linearly independent.

If in a set of vectors, any vector of the set is the combination of the remaining vectors, then the vectors are called dependent vectors.

**Example:**

1. Examine the following vectors for linear dependence and find the relation if it exists.

   $X_1 = (1, 2, 4), \ X_2 = (2, -1, 3), \ X_3 = (0, 1, 2), \ X_4 = (-3, 7, 2)$

**Solution.** Consider the matrix equation

$$\lambda_1 X_1 + \lambda_2 X_2 + \lambda_3 X_3 + \lambda_4 X_4 = 0$$

$$\lambda_1 (1, 2, 4) + \lambda_2 (2, -1, 3) + \lambda_3 (0, 1, 2) + \lambda_4 (-3, 7, 2) = 0 \quad \text{-------(1)}$$

$$\lambda_1 + 2\lambda_2 + 0\lambda_3 - 3\lambda_4 = 0$$

$$2\lambda_1 - \lambda_2 + \lambda_3 + 7\lambda_4 = 0$$

$$4\lambda_1 + 3\lambda_2 + 2\lambda_3 + 2\lambda_4 = 0$$

This is the homogeneous system

$$\begin{bmatrix} 1 & 2 & 0 & -3 \\ 2 & -1 & 1 & 7 \\ 4 & 3 & 2 & 2 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$R_2 \rightarrow R_2 - 2R_2$$

$$R_3 \rightarrow R_3 - 4R_1$$

$$\begin{bmatrix} 1 & 2 & 0 & -3 \\ 0 & -5 & 1 & 13 \\ 0 & -5 & 2 & 14 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_3 \rightarrow R_3 - R_2$

$$\begin{bmatrix} 1 & 2 & 0 & -3 \\ 0 & -5 & 1 & 13 \\ 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\lambda_1 + 2\lambda_2 - 3\lambda_4 = 0$$

$$-5\lambda_2 + \lambda_3 + 13\lambda_4 = 0$$

$$\lambda_3 + \lambda_4 = 0$$

Let $\lambda_4 = t$, $\lambda_3 + t = 0$, $\lambda_3 = -t$

$$-5\lambda_2 - t + \lambda_3 t = 0; \quad \lambda_2 = 12t/5$$

$$\lambda_1 + 24t/5 - 3t = 0; \quad \lambda_1 = -9t/5$$

Hence, the given vectors are linearly dependent.

Substituting the values of $\lambda$ in (1), we get

$$9\frac{tx_1}{5} + \frac{12t}{5}x_2 - tx_3 + tx_4 = 0$$

$$-\frac{9}{5}x_1 + \frac{12}{5}x_2 - x_3 + x_4 = 0$$

$$9x_1 - 12x_2 + 5x_3 - 5x_4 = 0$$

## 1.4 Inner Product (Scalar Product):

In ordinary three dimensional space the scalar product achieves,

1. The scalar product of a vector with itself helps to define the length of the vector.
2. It is a measure of relative orientation of the vectors, when the lengths are known.

In a linear vector space the inner product of two vectors ψ and φ is denoted by (ψ, φ). The inner product has the following properties.

**1.5 Properties of Inner Product:**

1. $(\psi, \varphi + \xi) = (\psi, \varphi) + (\psi, \xi)$
2. $(\psi + \varphi, \xi) = (\psi, \xi) + (\varphi, \xi)$
3. $(\psi, \psi) > 0$ unless $\psi = 0$
4. $(\psi, \varphi) = (\varphi, \psi)^*$
5. $(\psi, \alpha\varphi + \beta\xi) = \alpha(\psi, \varphi) + \beta(\psi, \xi)$ Where $\alpha$ and $\beta$ are arbitrary complex numbers.
6. The norm (length) is denoted by $\|\psi\|$, and is defined as $\|\psi\| = (\psi, \psi)1/2$

In an n- dimensional space, elements of basis are $\alpha_1, \alpha_2, \ldots\ldots \alpha n$, (the magnitude of each element of the basis is unity then the elements are called unit vectors) then two vectors $\psi$ and $\varphi$ in the space can be expressed as

$$\Psi = c_1\alpha_1 + c_2\alpha_2 + \ldots\ldots\ldots\ldots + c_n\alpha_n \text{ and}$$

$$\varphi = b_1\alpha_1 + b_2\alpha_2 + \ldots\ldots\ldots\ldots + b_n\alpha_n$$

Then the inner product of $\psi$ and $\varphi$ is

$$(\Psi, \varphi) = \sum_{i=1}^{n} c_i^* b_i = c_1^* b_1 + c_2^* b_2 + \ldots\ldots\ldots\ldots + c_n^* b_n$$

**Example**

1. Calculate inner product of the two vectors A and B given by

$$A = 5\alpha_1 - 3\alpha_2 - 4\alpha_3 - \alpha_4 + 2\alpha_5 \quad \text{and}$$

$$B = -\alpha_1 + 2\alpha_2 - 3\alpha_3 + \alpha_4 + \alpha_5$$

**Solution:**

The inner product of A and B is

$$(A, B) = (5)(-1) + (-3)(2) + (-4)(-3) + (-1)(1) + (2)(1)$$

$$= -5 - 6 + 12 - 1$$

$$= 2$$

2. Find the norm of a vector $3i + 4j + 5k$

**Solution:**

Let the vector $\psi = 3i + 4j + 5k$,

Then $(\psi, \psi) = (3i + 4j + 5k)\cdot(3i + 4j + 5k)$

$= 9 + 16 + 25$

$= 50$

$\|\psi\| = (\psi, \psi)1\ /2 = (50)1/2$

## 1.6 Orthogonality:

Two vectors x , y in $R_n$ are orthogonal or perpendicular if $x \cdot y = 0$. Notation: $x \perp y$ means

$$x \cdot y = 0.$$

**Orthogonal of Unit vector:**

Number of vectors that are mutually perpendicular to each other, meaning they form an angle of 90° with a magnitude of one unit with each other, are called orthogonal unit vectors.

## 1.7 The Gram-Schmidt orthogonalization process:

Let V be a vector space with an inner product. Suppose $x_1, x_2, \ldots, x_n$ is a basis for V.

Let $v_1 = x_1,$

$v_2 = x_2 - ( < x_2, v_1 > / < v_1, v_1 > ) v_1,$

$v_3 = x_3 - ( < x_3, v_1 > / < v_1, v_1 > ) v_1 - ( < x_3, v_2 > /< v_2, v_2 > ) v_2,$

.

.

.

.

$v_n = x_n - (< x_n, v_1 > /<v_1, v_1> ) v_1 - \cdot\cdot\cdot - ( < x_n, v_n > /< v_n, v_n> ) v_n,< v_{n-1}, v_{n-1} > v_{n-1}.$

Then $v_1, v_2, \ldots, v_n$ is an orthogonal basis for V.

## 1.8 Properties of the Gram-Schmidt process:

- $v_k = x_k - (\alpha_1 x_1 + \cdot\cdot\cdot + \alpha_{k-1} x_{k-1}),\ 1 \le k \le n;$
- the span of $v_1, \ldots, v_k$ is the same as the span of $x_1, \ldots, x_k$ ;
- $v_k$ is orthogonal to $x_1, \ldots, x_{k-1}$;
- $v_k = x_k - p_k$ , where $p_k$ is the orthogonal projection of the vector $x_k$ on the subspace spanned by $x_1, \ldots, x_{k-1}$;
- $v_k$ is the distance from $x_k$ to the subspace spanned by $x_1, \ldots, x_{k-1}$.

An alternative form of the Gram-Schmidt process combines orthogonalization with normalization. Suppose $x_1, x_2, \ldots, x_n$ is a basis for an inner product space V.

Let $v_1 = x_1$, $\qquad w_1 = \frac{V_1}{|V_1|}$

$v_2 = x_2 - \langle x_2, w_1\rangle\, w_1$, $\qquad w_2 = \frac{V_2}{|V_2|}$

$v_3 = x_3 - \langle x_3, w_1\rangle\, w_1 - \langle x_3, w_2\rangle\, w_2$, $\qquad w_3 = \frac{V_3}{|V_3|}$

.

.

.

$v_n = x_n - \langle x_n, w_1\rangle w_1 - \cdots - \langle x_n, w_{n-1}\rangle w_{n-1}$, $\quad w_n = v_n/|v_n|$ .

Then $w_1, w_2, \ldots, w_n$ is an orthonormal basis for V.

**Example**

Let Π be the plane in $R_3$ spanned by vectors $x_1 = (1, 2, 2)$ and $x_2 = (-1, 0, 2)$.

(i) find orthonormal basis for Π.

(ii) Extend it to an orthonormal basis for $R_3$ . $x_1, x_2$ is a basis for the plane Π.

**Solution:**

We can extend it to a basis for $R_3$ by adding one vector from the standard basis. For instance, vectors $x_1$, $x_2$, and $x_3 = (0, 0, 1)$ form a basis for $R_3$ because

$$\begin{bmatrix} 1 & 2 & 2 \\ -1 & 0 & 2 \\ 0 & 0 & 1 \end{bmatrix} = \begin{vmatrix} 1 & 2 \\ -1 & 0 \end{vmatrix} = 2 \text{ (not equal to zero)}$$

Using the Gram-Schmidt process, we orthogonalize the basis

$x_1 = (1, 2, 2)$,

$x_2 = (-1, 0, 2)$,

$x_3 = (0, 0, 1)$

$v_1 = x_1 = (1, 2, 2)$,

$v_2 = x_2 - (\langle x_2, v_1\rangle / \langle v_1, v_1\rangle)\, v_1$

$= (-1, 0, 2) - 3\ 9\ (1, 2, 2)$

$= (-4/3, -2/3, 4/3)$,

$v_3 = x_3 - (\langle x_3, v_1\rangle/\langle v_1, v_1\rangle)\, v_1 - (\langle x_3, v_2\rangle/\langle v_2, v_2\rangle)\, v_2$

$= (0, 0, 1) - 2\ 9\ (1, 2, 2) - 4/3\ 4\ (-4/3, -2/3, 4/3)$

$= (2/9, \ -2/9, \ 1/9)$.

Now $v_1 = (1, 2, 2)$,

$v_2 = (-4/3, -2/3, 4/3)$,

$v_3 = (2/9, -2/9, 1/9)$ is an orthogonal basis for $R_3$ while $v_1$, $v_2$ is an orthogonal basis for Π.

It remains to normalize these vectors.

$< v_1, v_1 > = 9 = \Rightarrow |v_1| = 3$

$< v_2, v_2 > = 4 = \Rightarrow |v_2| = 2$

$< v_3, v_3 > = 1/9 = \Rightarrow |v_3| = 1/3$

$W_1 = v_1/|v_1| = (1/3, 2/3, 2/3) = 1/3\ (1, 2, 2)$,

$w_2 = v_2/|v_2| = (-2/3, -1/3, 2/3) = 1/3\ (-2, -1, 2)$,

$w_3 = v_3/|v_3| = (2/3, -2/3, 1/3) = 1/3\ (2, -2, 1)$.

$w_1$, $w_2$ is an orthonormal basis for Π. $w_1$, $w_2$, $w_3$ is an orthonormal basis for $R_3$ .

## 1.9 Linear Operator:

A linear operator is a function that maps one vector onto other vectors. They can be represented by matrices, which can be thought of as coordinate representations of linear operators ( Hjortso & Wolenski, 2008). Therefore, any *n* x *m* matrix is an example of a linear operator.

A linear operator is usually (but not always) defined to satisfy the conditions of additivity and multiplicativity.

- Additivity: $f(x + y) = f(x) + f(y)$ for all x and y,
- Multiplicativity: $f(cx) = cf(x)$ for all x and all constants c.

## 1.10 Dual space:

All linear transformations from one vectors space *V* into another vector space *W* is denoted as V(*V*, *W*). Its particular case arises when we choose $W = \mathbb{F}$ (as a one-dimensional coordinate vector space over itself.

Let $V$ = C be the set of all complex numbers over the field C (itself). We consider the involution operation

$$J: C \mapsto C \qquad J(x + jy) = z^* = x - jy.$$

So $J$ maps the complex plane into itself by swapping it with respect to the abscissa (called the real axis in C). Note that we denote complex conjugate by $z^*$ instead of over line notation

$$\bar{z} = \overline{x + \jmath y} = x - jy$$

As you see, the asterisk notation is in agreement with notation of dual spaces.

When $V$ is considered as a complex vector space, then complex conjugate convolution is not a linear operation because

$$J(c\,z) = c{*}z{*}, \quad c \in C.$$

However, when $V = \mathbb{C}$ is considered as a vector space over the field of real numbers, $J$ is a linear transformation.

**1.11<u>Basis:</u>**

Let V be a vector space. A collection of vectors $\alpha_1, \alpha_2, \alpha_3, \ldots\ldots\ldots \alpha_r$ is said to form a basis of V if $\alpha_1, \alpha_2, \alpha_3, \ldots\ldots\ldots \alpha_r$ linearly independent and if they generate V.

**<u>Change of basis:</u>**

Here we will illustrate the change in O basis by the following:

Let the coordinate of the point A be (4, 5).

If two persons X and Y want to go from O to A.

Person X starts from O and uses the path OB and BA accordingly he has to go first four steps along OX and then five steps parallel to y – axis to reach A.

Person Y starts from o and uses the path OC and then CA accordingly he has to go first 5 steps along OY – axis and then 4 steps parallel to X axis to reach A.

Here both the person reach A by using different paths.

The only difference is in change of order of the paths. These paths are expressed in the standard basis of $R^2$.

$$S_1 = \{(1,0), (0,1)\}$$

$$S_1 = \{(0,1), (1,0)\}$$

Path 1 = (4, 5) = 4 (1,0) + 5(0,1)

Path 2 = (4, 5) = 5 (0,1) + 4(1,0)

In path 1 and 2 the coefficients are actually directions in terms of the standard basis of $R^2$. now, the vector order is important in a basis. Here the direction vectors or coordinates vectors are $\begin{bmatrix}4\\5\end{bmatrix}$ and $\begin{bmatrix}5\\4\end{bmatrix}$

The vector can be expressed in form of matrix are $\begin{bmatrix}4\\5\end{bmatrix} = \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}\begin{bmatrix}4\\5\end{bmatrix}$

$$\begin{bmatrix}4\\5\end{bmatrix} = \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}\begin{bmatrix}5\\4\end{bmatrix}$$

We can express any vector by using standard basis easily, but we have to describe the given vector in term of different basis.

**Example:**

Let X = ( 4, 5) in $R^2$, find the coordinate vector for X with respect to the basis

A = { (1,1) ,(-1,2)}

**Solution:**

$$(4, 5) = C_1\,(1,1) + C_2\,(-1,2)$$

$$= (C_1, C_1) + (-C_2, 2C_2)$$

$$= (C_1 - C_2\ ,C_1 + 2C_2\ )$$

Here, $C_1 - C_2\ = 4\ C_1 + 2C_2 = 5$

On solving, we get $C_1 = \frac{13}{3}, C_2\ = \frac{1}{3}$

The coordinate vector for X with respect to A is $[X]_A = \left(\frac{13}{3}, \frac{1}{3}\right)$

## 1.12 Isomorphism of vector space

Two vector spaces V and W over the same field F are isomorphic if there is a bijection T : V → W which preserves addition and scalar multiplication, that is, for all vectors u and v in V , and all scalars

c ∈ F, T (u + v) = T (u) + T (v) and

T (cv) = c T(v).

The correspondence T is called an isomorphism of vector spaces. When T : V → W is an isomorphism we'll write T : V '→ W if we want to emphasize that it is an isomorphism. When V and W are isomorphic, but the specific isomorphism is not named, we'll just write V ~= W.

Of course, the identity function IV : V '→ V is an isomorphism. After we introduce linear transformations (which is what homomorphisms of vector spaces are called), we'll have another way to describe isomorphisms.

Since the structure of vector spaces is defined in terms of addition and scalar multiplication, if T preserves them, it will preserve structure defined in terms of them. For instance, T preserves 0, negation, subtraction, and linear transformations.

**1.13 Projection Operator:**

Suppose that **A** is an $m \times n$ real matrix of rank $n$ (full column rank). Then the matrix

$\mathbf{P} = \mathbf{A}\,(\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T$ represents the orthogonal projection of $R^m$ onto the range of **A** (span of the column space).

If **A** is not a full column rank matrix, then $A^TA$ is not invertible. Thus, we get another criterion for consistency of the linear algebraic equation Ax = b:

$$Pb = b,$$

where **P** is the **projection operator** on the column space of an m×n real matrix.

## 1.14 Eigen values and Eigen functions:

Let , $AX = Y$ ------------(1)

Where A is the matrix, X is the column vector and Y is also column vector.

Here column vector X is transformed into the column vector Y by means of the square matrix A.

Let x be a such vector which transforms into $\lambda X$ by means of the transformation (1). Suppose, the linear transformation Y = AX transforms X into a scalar multiple of itself i.e., $\lambda X$.

$$AX = Y = \lambda X$$

$$AX - \lambda IX = 0$$

$$(A - \lambda I)X = 0 \quad \text{---------------(2)}$$

Thus unknown scalar $\lambda$ is known as eigen value of the matrix A and the corresponding non zero vector X as eigen vector.

## 1.15 Direct sum of vector sub – space:

A vector space V is said to be the direct sum of two of its sub space $w_1$ and $w_2$ , written as V = $w_1$ , if each element of V is uniquely expressible as sum of an element of $w_1$ and an element of $w_2$

In this case, $w_1$ and $w_2$ are said to be complementary sub space.

The above definition can be extended for more than two sub space.

Thus, $V = w_1 \oplus w_2 \oplus$ .................... $w_n$ each $\alpha \in V$ is uniquely expressible as

$$\alpha = \alpha_1 + \alpha_2 + \ldots\ldots\ldots\ldots + \alpha_n$$

Where $\alpha_I \in W_I$ for each i = 1, 2,…..n

The following theorem gives criteria for a vector space to be the direct sum of two of its sub space.

**Theorem:**

The necessary and sufficient conditions for a vector space V (F) to be the direct sum of its subspace $w_1$ and $w_2$ are

(i) $V = w_1 + w_2$

(ii) $w_1 \cap w_2 = (0)$

## 1.16 Orthogonal transformation:

A transformation Y = AX is said to be orthogonal if its matrix is orthogonal.

**Theorem:**

A linear transformation preserves lengths, if it preserves inner product.

**Proof:**

Let the linear transformation is

$$Y = AX$$

Let $X_1$ and $X_2$ be any two vectors whose images are $Y_1$ and $Y_2$ respectively.

$$Y_1 = AX_1$$

$$Y_2 = AX_2$$

Now $|X_1^2 + X_2^2|^2 = |X_1^2| + |X_2^2| + 2. X_1 X_2$

$$X_1 X_2 = \frac{1}{2} [\, |X_1^2 + X_2^2|\text{^}2 - |x_1|^2 - |x_2|^2 \,]$$

Similarly, $Y_1 Y_2 = \frac{1}{2} [\, |Y_1^2 + Y_2^2|\text{^}2 - |y_1|^2 - |y_2|^2 \,]$

From the above results, the given linear transformation preserves length if and only if it preserves the inner product.

**Theorem:**

A linear transformation preserves lengths if and only if its matrix is orthogonal.

**Proof:**

Let $X_1$ and $X_2$ be two vectors and $Y_1$ , $Y_2$ be their images respectively

$$Y_1 = AX_1$$

$$Y_2 = AX_2$$

$$(Y_1 . Y_2) = Y_1^T Y_2 = (AX_1)^T(AX_2)$$

$$= X_1^T A^T AX_2$$

$$= X_1^T I X_2$$

$$= X_1^T X_2$$

$$= (X_1 , X_2)$$

Thus, the linear transformation preserves the inner product and therefore it preserves length.

## UNITII: COMPLEX ANALYSIS, PROBABILITY&STATISTICS

Review of Complex Numbers - de Moivre's Theorem-Functions of a Complex Variable-Differentiability - Analytic functions- Harmonic Functions - Complex Integration- Contour Integration, Cauchy – Riemann conditions – Singular points – Cauchy's Integral Theorem and integral Formula -Taylor's Series - Laurent's Expansion - Zeros and poles – Residue theorem. Probability – Introduction – Addition rule of probability – Multiplication law of probability– Problems – Introduction to statistics – Mean, median, mode and standard deviations.

### 2.1 Complex Number:

A number of the form $a + i\,b$ is called a complex number when $a$ and $b$ are real numbers and $i = -1$ . We call '$a$' the real part and '$b$' the imaginary part of the complex number $a + ib$. If $a = 0$ the number $i\,b$ is said to be purely imaginary, if $b = 0$ the number $a$ is real.

A complex number $x + iy$ is denoted by $z$.

### 2.2 DE MOIVRE'S THEOREM (By Exponential Function)

$$(\cos\theta + i\sin\theta)n = \cos n\theta + i\sin n\theta$$

**Proof.**

We know that $e^{i\theta} = \cos\theta + i\sin\theta$

$$(e^{i\theta}) = (\cos\theta + i\sin\theta)$$

$$e^{in\theta} = (\cos\theta + i\sin\theta)^n$$

$$(\cos n\theta + i\sin n\theta) = (\cos\theta + i\sin\theta)^n$$ **Proved.**

If $n$ is a fraction, then $\cos n\theta + i\sin n\theta$ is one of the values of $(\cos\theta + i\sin\theta)$

### 2.3 DE MOIVRE'S THEOREM (BY INDUCTION)

**Statement:**

For any rational number $n$ the value or one of the values of

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$$

**Case I:**

Let $n$ be a non-negative integer. By actual multiplication,

$$(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2) = (\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i(\cos\theta_1\sin\theta_2 + \sin\theta_1\cos\theta_2)$$

$$= \cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)$$

Similarly we can prove that

$$(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2)(\cos\theta_3 + i\sin\theta_3) = \cos(\theta_1 + \theta_2 + \theta_3) + i\sin(\theta_1 + \theta_2 + \theta_3)$$

Continuing in this way, we can prove that

$$(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2)\ldots(\cos\theta_n + i\sin\theta_n) = \cos(\theta_1 + \theta_2 \ldots + \theta_n) + i\sin(\theta_1 + \theta_2 + \ldots + \theta_n)$$

Putting $\theta_1 = \theta_2 = \theta_3 = \ldots \theta_n = \theta$, we get

$$(\cos\theta + i\sin\theta)^n = (\cos n\theta + i\sin n\theta)$$

**Case II:**

Let $n$ be a negative integer, say $n = -m$ where $m$ is a positive integer. Then,

$$(\cos\theta + i\sin\theta)^n = (\cos\theta + i\sin\theta)^{-m}$$

$$= 1/(\cos\theta + i\sin\theta)^m$$

$$= 1/(\cos m\theta + i\sin m\theta) \qquad \text{[By case I]}$$

$$= (1/(\cos m\theta + i\sin m\theta)).((\cos m\theta - i\sin m\theta)/(\cos m\theta - i\sin m\theta))$$

$$= (\cos m\theta - i\sin m\theta)/(\cos^2 m\theta + \sin^2 m\theta)$$

$$= \cos m\theta - i\sin m\theta$$

$$= \cos(-m\theta) + i\sin(-m\theta)$$

$$= \cos n\theta + i\sin n\theta$$

Hence, the theorem is true for negative integers also.

**Case III:**

Let $n$ be a proper fraction $p/q$. Where $p$ and $q$ are integers. Without loss of generality we can select $q$ to be positive integer, $p$ may be a positive or negative integer. Since $q$ is a positive integer.

$$\left(\cos\frac{\theta}{q} + i\sin\frac{\theta}{q}\right)^q = \cos q\cdot\frac{\theta}{q} + i\sin q\cdot\frac{\theta}{q}$$

$$= \cos\theta + i\sin\theta$$

Taking the $q^{th}$ root of both sides, we get

$$(\cos\theta + i\sin\theta)^{1/q} = \cos\frac{\theta}{q} + i\sin\frac{\theta}{q}$$

Raising both sides to the power *p*,

$$(\cos\theta + i\sin\theta)^{p/q} = \cos\frac{\theta}{q} + i\sin\frac{\theta}{q}$$

$$= \cos p\cdot\frac{\theta}{q} + i\sin p\cdot\frac{\theta}{q} \qquad \text{[By case I and II]}$$

Hence, one of the values of $(\cos\theta + i\sin\theta)^n$ is $\cos n\theta + i\sin n\theta$ when n is a proper fraction.

Thus, the theorem is true for all rational values of *n*.

**Example:**

Express

$\dfrac{(\cos\theta+i\sin\theta)^8}{(\sin\theta+i\cos\theta)^4}$ in the form (x + iy).

**Solution:**

$$\frac{(\cos\theta+i\sin\theta)^8}{(\sin\theta+i\cos\theta)^4} = \frac{(\cos\theta+i\sin\theta)^8}{i^4\left(\sin\theta+\frac{1}{i}\cos\theta\right)^4}$$

$$= \frac{(\cos\theta+i\sin\theta)^8}{(\cos\theta-i\sin\theta)^4}$$

$$= \frac{(\cos\theta+i\sin\theta)^8}{(\cos(-\theta)+i\sin(-\theta))^4}$$

$$= \frac{(\cos\theta+i\sin\theta)^8}{[(\sin\theta+i\cos\theta)^{-1}]^4} = \frac{(\cos\theta+i\sin\theta)^8}{(\cos\theta+i\sin\theta)^4}$$

$$= (\cos\theta + i\sin\theta)^{12}$$

$$= \cos 12\theta + i\sin 12\theta$$

## 2.4 Function of complex variables:

The theory of functions of a complex variable is of atmost importance in solving a large number of problems in the field of engineering and science. Many complicated integrals of real functions are solved with the help of functions of a complex variable.

$x + iy$ is a complex variable and it is denoted by $z$.

(1) $z = x + iy$ where $i = -1$ (Cartesian form)

(2) $z = r(\cos\theta + i\sin\theta)$ (Polar form)

(3) $z = re^{i\theta}$ (Exponential form)

$f(z)$ is a function of a complex variable $z$ and is denoted by $w$.

$$w = f(z)$$

$$w = u + iv$$

where $u$ and $v$ are the real and imaginary parts of $f(z)$.

**2.5 Differentiability**:

Let $f(z)$ be a single valued function of the variable $z$, then

$$f'(z) = \lim_{\delta z \to 0} \frac{f(2+\delta z) - f(z)}{\delta z}$$

provided that the limit exists and is independent of the path along which dz → 0.

Let $P$ be a fixed point and $Q$ be a neighboring point. The point $Q$ may approach $P$ along any straight line or curved path.

**Example:**

Consider the function

f (z) = 4x + y + i(–x + 4y) and discuss df/dz

**Solution:**

here, $f(z) = 4x + y + i(-x + 4y) = u + iv$

so $u = 4x + y$ and $v = - x + 4y$

$f(z + \delta z) = 4(x + \delta x) + (y + \delta y) \quad i(x + \delta x) + 4i(y + \delta y)$

$f(z + \delta z) - f(z) = 4(x + \delta x) + (y + \delta y) - i(x + \delta x) + 4i(y + \delta y) - 4x - y + ix - 4iy$

$$= 4\delta x + \delta y - i\,\delta x + 4i\delta y$$

$\frac{f(z+\delta z)-f(z)}{\delta z}$ = $(4\delta x + dy - i\,dx + 4idy) / (dx + i\,dy)$

($a$) **Along real axis:** If $Q$ is taken on the horizontal line through $P(x, y)$ and $Q$ then approaches $P$ along this line, we shall have $dy = 0$ and $dz = dx$

$$\frac{\delta f}{\delta z} = \frac{4\delta x - i\delta x}{\delta x} = 4 - i$$

($b$) **Along imaginary axis:** If $Q$ is taken on the vertical line through $P$ and then $Q$ approaches $P$ along this line, we have

$$z = x + iy = 0 + iy, \quad dz = i dy, \quad dx = 0$$

Putting these values in (1), we have

$$\frac{\delta f}{\delta z} = \frac{\delta y + 4i\delta y}{i\delta y} = i^{-1}(1+4i) = 4-i$$

($c$) **Along a line $y = x$ :** If $Q$ is taken on a line $y = x$. $z = x + iy = x + ix = (1 + i)x$

$dz = (1 + i)\, dx$ and $dy = dx$

On putting these values in (1), we have

$$\frac{\delta f}{\delta z} = \frac{4\delta x + \delta x - idx + 4idx}{\delta x + idx}$$

$$= \frac{4+1-i+4i}{1+i}$$

$$= \frac{5+3i}{1+i}$$

$$= \frac{5+3i}{1+i}\ \frac{1-i}{1-i}$$

$$= 4-i$$

In all the three different paths approaching $Q$ from $P$, we get the same values of $\frac{\delta f}{\delta z} = 4-i$

In such a case, the function is said to be differentiable at the point $z$ in the given region.

## 2.6 Analytic function:

A function $f(z)$ is said to be **analytic** at a point $z_0$, if $f$ is differentiable not only at $z_0$ but at every point of some neighbourhood of $z_0$.

A function $f(z)$ is analytic in a domain if it is **analytic** at every point of the domain. The point at which the function is not differentiable is called a **singular point** of the function. An analytic function is also known as "holomorphic", "regular", "monogenic".

**Entire Function.** A function which is analytic everywhere (for all $z$ in the complex plane) is known as an entire function.

**For Example**

**1**. Polynomials rational functions are entire

**2.** $|z|^2$ is differentiable only at $z = 0$. So it is no where analytic.

**Note:** ($i$) An entire is always analytic, differentiable and continuous function. But convers is not true.

(*ii*) Analytic function is always differentiable and continuous. But converse is not true.

(*iii*) A differentiable function is always continuous. But converse is not true

## THE NECESSARY CONDITION FOR F (Z) TO BE ANALYTIC

**Theorem.** The necessary conditions for a function f (z) = u + iv to be analytic at all the points in a region R are

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$, $\frac{\partial v}{\partial x}$, $\frac{\partial v}{\partial y}$ are continuous function.

**Example:**

Determine whether 1/z is analytic or not?

**Solution:**

Let $$w = f(z) = u + iv = 1/z$$

$$u+iv = 1/(x+iy)$$

$$= \frac{x-iy}{x^2+y^2}$$

Equating real and imaginary parts, we get

$$u = \frac{x}{x^2+y^2} \qquad v= \frac{-y}{x^2+y^2}$$

$$\frac{\partial u}{\partial x} = \frac{(x^2+y^2).1-x.2x}{(x^2+y^2)^2} \qquad = \frac{y-x^2}{(x^2+y^2)^2}$$

$$\frac{\partial u}{\partial y} = \frac{-2xy}{(x^2+y^2)^2}$$

$$\frac{\partial v}{\partial x} = \frac{2xy}{(x^2+y^2)^2}$$

$$\frac{\partial v}{\partial y} = \frac{(y^2-x^2)}{(x^2+y^2)^2}$$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Thus C – R equations are satisfied. Also partial derivatives are continuous except at (0, 0).

Therefore1/*z* is analytic everywhere except at *z* = 0.

also $$\frac{dw}{dz} = -\frac{1}{z^2}$$

This again shows that *dw/dz* exists everywhere except at *z* = 0. Hence 1/*z* is analytic everywhere except at *z* = 0.

## 2.7 HARMONIC FUNCTION

Any function which satisfies the Laplace's equation is known as a harmonic function.

**Theorem.**

If *f* (*z*) = *u* + *iv* is an analytic function, then *u* and *v* are both harmonic functions.

**Proof.**

Let *f* (*z*) = *u* + *iv*, be an analytic function, then we have

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \ldots\ldots\ldots\ldots\ldots\ldots(1)$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \quad \ldots\ldots\ldots\ldots\ldots(2)$$

Differentiating (1) with respect to *x*, we get $$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x \partial y} \quad \ldots\ldots\ldots\ldots\ldots..(3)$$

Differentiating (2) w.r.t. '$y$' we have $$\frac{\partial^2 u}{\partial y^2} = -\frac{\partial^2 v}{\partial y \partial x} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots.(4)$$

Adding (3) and (4) we have $$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 v}{\partial x \partial y} - \frac{\partial^2 v}{\partial y \partial x}$$

$$\frac{\partial^2 u}{\partial x} + \frac{\partial^2 u}{\partial y} = 0$$

Similarly $$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$$

Therefore both *u* and *v* are harmonic functions.

Such functions *u*, *v* are called **Conjugate harmonic functions if *u* + *iv* is also analytic function.**

## 2.8 Complex integration and counter integration:

In case of real variable, the path of integration of $\int_a^b f(x)dx$ is always along the x-axis from x = a to x = b. but in case of complex function f (z) the path of the definite integral $\int_a^b f(z)dz$ can be along any curve from z = a to z = b.

z = x + iy

dz = dx + i dy -----------(1)

dz = dx if y = 0 -----------(2)

dz = i dy if x = 0 ------------(3)

in equation (1),(2) (3) the direction of dz are different.

Its value depends upon the path (curve) of integration. But the value of integral from a to b remains the same along any regular curve from a to b.

In case the initial point and final point coincide so that c is closed curve, then this integral is called **contour integral** and is denoted by $\oint_c f(z)dz$.

If f(z) = u(x,y) + iv (x,y), then since dz = dx + idy

$$\oint_c f(z)dz = \int_c (u + iv)(dx + dy)$$

$$= \int_c (u\,dx - vdy) + \int_c (vdx + u\,dy)$$

Which shows that the evaluation of the line integral of a complex function can be reduced to the evaluation of two line integrals of real functions.

## 2.9 Cauchy's Reiman function:

Let f(x,y) = u(x,y) + iv (x,y) ----------------(1)

Where z = x + iy ----------------(2)

So dz = dx + i dy -----------------(3)

The total derivative of f with respect to z is then

$$\frac{df}{dz} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial z} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial z} \quad \text{------------------(4)}$$

$$= \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right) \quad \text{------------------(5)}$$

In terms of u and v, (5) becomes

$$\frac{\partial f}{\partial x} = \frac{1}{2}\left(\left(\frac{\partial u}{\partial x} + i\,\frac{\partial v}{\partial x}\right) - i\left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right)\right) \qquad \text{-----------(6)}$$

$$= \frac{1}{2}\left(\left(\frac{\partial u}{\partial x} + i\,\frac{\partial v}{\partial x}\right) + \left(-i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}\right)\right) \qquad \text{-----------------(7)}$$

Along the real or x – axis $\frac{\partial f}{\partial y} = 0$. So

$$\frac{\partial f}{\partial z} = \frac{1}{2}\left(\frac{\partial u}{\partial x} + i\,\frac{\partial v}{\partial x}\right) \qquad \text{--------------(8)}$$

along the imaginary, or y − axis $\frac{\partial f}{\partial x} = 0$, so

$$\frac{\partial f}{\partial z} = \frac{1}{2}\left(-i\,\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}\right) \qquad \text{-----------(9)}$$

If f is complex differentiable, then the value of the derivative must be same for a given dz, regardless of its orientation. Therefore, (8) must equal (9)

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \text{-----------(10)}$$

and $$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \qquad \text{--------------(11)}$$

these are known as the Cauchy - Reimann equations.

## 2.10 Cauchy's integral theorem:

**Statement:**

If a function $(z)$ is analytic and $f'(z)$ is continuous at every point inside and on a simple closed curve $C$, then $\int_c z dz = 0$.

**Proof :**

Let the region enclosed by the curve $R$ and

$(z) = u\, x, y + (x, y)$ with $z = x + iy \Longrightarrow dz = dx + idy$

$\int_c f(z)dz = \int_c u + iv\,(dx + idy)$

$\int_c (udx - vdy) + i \int_c (vdx + udy)$

*Since* $f'(z)$ *is continuous* $,\partial u/\partial x,\ \partial u/\partial y,\ \partial v/\partial x,\ \partial v/\partial y$ *are also continuous in* $R$

By applying Green's theorem,

$$\int_c (Pdx + Qdy) = \int\int_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{dy}\right) dx\, dy \text{ in each integral}$$

*we obtain*

$$\int_c (udx - vdy) = \int\int_R \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{dy}\right) dx\, dy$$

$$\int_c (vdx + udy) = \int\int_R \left(\frac{\partial u}{\partial x} - \frac{\partial v}{dy}\right) dx\, dy$$

$$\int_c f(z)dz = \int\int_R \left(\frac{\partial u}{\partial x} - \frac{\partial v}{dy}\right) dx\, dy + i \int\int_R \left(\frac{\partial u}{\partial x} - \frac{\partial v}{dy}\right) dx\, dy$$

Since $(z)$ is analytic, $u$ and $v$ satisfy CR- equations so the integrands of the two integral in right hand side of above equation vanishes and we get,

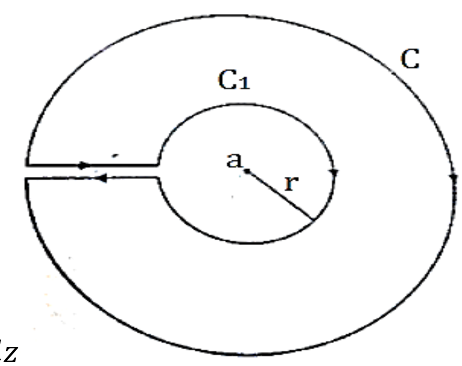
$$\int_c f(z)dz = 0$$

Hence, proved the theorem.

## 2.10 Cauchy's Integral Formula:

**Statement:**

If a function $(z)$ is analytic within on a closed curve $C$ and if $a$ is any point inside $c$ then




$$f(a) = \frac{1}{2\pi i}\int \frac{f(z)}{z-a} dz$$

**Proof:**

Let us consider the function $(z)/z -$ which is analytic at all points inside $C$, except at $z = a$.

With point $a$ as center and radius $r$, draw a small circle $C_1$ lying completely within $C$. (Figure)

Since$(z)/\ z-a$ is analytic in the region between $C$and $C_1$, we have by Cauchy's theorem

Since for any point $C_1$ , $z-a=re^{i\theta} \Longrightarrow dz= ire^{i\theta}d\theta$

$$\int_c \frac{f(z)}{z-a} = \int_{c_1} \frac{f(z)}{z-a}dz$$

$$= \int_{c_1} \frac{f(a+re^{i\theta})}{re^{i\theta}}\ ire^{i\theta}d\theta$$

$$= i\int_{c_1} \frac{f(a+re^{i\theta})}{1}\ d\theta$$

In the limit $C_1$ shrinks to point $a$ ie., as $r \to 0$ *the integral approaches*

This is Cauchy's integral formula.

**Example:**

*For a complex variable z, resolve ln z into real and imaginary parts.*

**Solution:**

$we\ have\ z = x + i, using\ x = rcos\theta\ and\ y = r\ sin\theta$

$$z = rcos\theta + rsin\theta \Longrightarrow r(cos\theta + isin\theta) \Longrightarrow r$$

Where $r = x_2 + y2$ and $\theta = \tan^{-1}$ (x/y)

$\therefore \ln z = \ln (\ re^{i\theta}\ ) = \ \ \ln (\ r) + \ln (e^{i\theta}) = \ln\sqrt{x^2+y^2} + i\ \theta$

$$= (1/2) \ln (x^2+y^2) + i \tan^{-1} (x/y)$$

Real part is (1/2) ( $x^2 + y^2$) and the imaginary part is $\tan^{-1}$ (y / x)

**Example:**

Using Cauchy-Riemann condition show that $W = sinz$ is analytic

**Solution:**

$$W = sinz \Longrightarrow \sin x + iy = sinx\ coshy + i\ cosx\ sinhy$$

*ie.* , $u = sinx\ coshy \Longrightarrow \partial u/\partial x = cosx\ coshy$ *and*

$$\partial u/\partial y = sinx\ sinhy$$

$$v = \cos x \sinh y \Rightarrow \partial v/\partial y = \cos x \cosh y \ and$$

$$\partial v/\partial x = -\sin x \sinh y$$

$it\ satisfies\ CR\ equations\ \partial u/\partial x = \partial v/\partial y$

$$\partial u/\partial y = -\partial v/\partial x$$

$\therefore W = \sin z\ is\ analytic.$

## 2.11 TAYLOR'S SERIES METHOD

Let us consider the first order differential equation

$$\frac{dy}{dx} = f(x, y)$$

under the condition $y = 0$ for $x = x_0$.

On differentiating (1) again and again, we get $\frac{d^2y}{dx^2}$ , $\frac{d^3y}{dx^3}$, $\frac{d^4y}{dx^4}$ etc.

On putting $x = x_0$ and $y = 0$ in the above equations we get the values of

$$\frac{dy}{dx}, \frac{d^2y}{dx^2}, \frac{d^3y}{dx^3}, \frac{d^4y}{dx^4}$$

substituting the values of $y', y'', y''', y''''\ldots$ in Taylor's series

$$y = y_0 + (x - x_0)[y'(x_0)] + \frac{(x-x_0)^2}{2!}[y''(x_0)] + \frac{(x-x_0)^3}{3!}[y''(x_0)] + \ldots\ldots$$

Thus we can obtain a power series for $y(x)$ in powers of $(x - x_0)$.

**Example:**

Using Taylor's series method, obtain the solution of dy/dx= 3 x + $y^2$ and

y = 1, when x = 0. Find the value of y for x = 0.1, correct to four places of decim*als.*

**Solution.**

$$\frac{dy}{dx} = 3x + y^2 \ldots\ldots\ldots\ldots\ldots\ldots(1)$$

$$y(0) = 1 \quad \ldots\ (2)$$

Differentiating (1) w.r.t '$x$', we get $\frac{d^2y}{dx^2} = 3 + 2y\frac{dy}{dx}$

$$\frac{d^3y}{dx^3} = 2y\frac{d^2y}{dx^2} + 2\left(\frac{dy}{dx}\right)^2$$

$$\frac{d^4y}{dx^4} = 2y\frac{d^3y}{dx^3} + 2\left(\frac{dy}{dx}\right)^1\frac{d^2y}{dx^2} + 4\left(\frac{dy}{dx}\right)^1\frac{d^2y}{dx^2}$$

$$\frac{dy}{dx} = 0 + (1)^2 = 1$$

$$\frac{d^2y}{dx^2} = 3 + 2\,(1)(1) \;= 5$$

$$\frac{d^3y}{dx^3} = 2(1)(5)^2_{2(1)} = 12$$

$$\frac{d^4y}{dx^4} = 2(1)(12) + 2(1)(5) + 4(1)(5) = 54$$

We know by Taylor's series expansion

$$y = y_0 + (x - x0)\,[y'\,(x0)] + \frac{(x-x_0)^2}{2!}[y''(x_0)] + \frac{(x-x_0)^3}{3!}[y''(x_0)] + \ldots\ldots$$

On substituting the value of $y$ (0), $y'$ (0), $y''$ (0), $y'''$ (0), $yiv$ (0) etc.

$$y = 1 + x + +\frac{(x)^2}{2!}[(5)] + \frac{x^3}{3!}[(12)] + +\frac{x^4}{4!}[(54)] + \ldots\ldots$$

$y$ (0.1) = 1 + 0.1 +(5/2)(0.01) + 2 (0.001) +(9/4) (0.0001) + …

= 1 + 0.1 + 0.025 + 0.002 + 0.000225

= 1.127225

## 2.12 Laurents's expansion:

If required to expand f (z) about a point where f (z) is not analytic, then it cam be expanded by Laurent series and not by Taylor's series.

**Statement:**

If f (z) is analytic on $c_1$ and $c_2$ , and the annular region R bounded by the two concentric circles $c_1$ and $c_2$ of radii $r_1$ and $r_2$ ($r_2 < r_1$) and with centre at a, then for all z in R

$$f(z) = a_0 + a_1(z - a) + a_2(z - a)^2 + \ldots\ldots + \frac{b_1}{z-a} + \frac{b_2}{(z-a)\hat{}2} + \ldots\ldots$$

where $\quad a_n = \frac{1}{2\pi i} \int_{c1} \frac{f(w)}{(w-a)^{n+1}} dw,$

$$b_n = \frac{1}{2\pi i} \int_{c2} \frac{f(w)}{(w-a)^{-n+1}} dw.$$

**Example:**

Expand $f(z) = \frac{1}{(z-1)(z-2)}$ for I < |z| < 2

**Solution:**

$$f(z) = \frac{1}{(z-1)(z-2)} = \frac{1}{(z-2)} - \frac{1}{(z-1)}$$

in first bracket |z| < 2, we take out 2 as common and from second bracket z is taken out common as 1 < |z|.

$$f(z) = -\frac{1}{2}\left(\frac{1}{1-\frac{z}{2}}\right) - \frac{1}{z}\left(\frac{1}{1-\frac{1}{z}}\right)$$

$$= -\frac{1}{2}\left(1-\frac{z}{2}\right)^{-1} - \frac{1}{z}\left(1-\frac{1}{z}\right)^{-1}$$

$$= -\frac{1}{2}\left[1+\frac{z^1}{2}+\frac{z^2}{4}+\frac{z^3}{8}+\cdots ....\right] - \frac{1}{z}\left[1+\frac{1}{z}+\frac{1}{z^2}+\frac{1}{z^3}+\cdots ....\right]$$

$$= -\frac{1}{2} - \frac{z^1}{4} - \frac{z^2}{8} - \frac{z^3}{16} + \cdots .... - \frac{1}{z} - \frac{1}{z^2} - \frac{1}{z^3} - \cdots ....$$

**Example :**

Find the Laurent series expansion of

$f(z) = \frac{1}{(z-1)(z-2)}$ valid for |z – 1| >1

**Solution:**

$$f(z) = \frac{1}{(z-1)(z-2)} = \frac{-1}{(z-2)} + \frac{2}{(z-1)}$$

$$= \frac{-1}{(z-1)} + \frac{2}{(z-1-1)}$$

$$= \frac{1}{(z-1)} + \frac{2}{(z-1)} \frac{1}{1-\frac{1}{z-1}} = \frac{1}{(z-1)} + \frac{2}{(z-1)} \left(1 - \frac{1}{z-1}\right)^{-1}$$

$$= \frac{1}{(z-1)} + \frac{2}{(z-1)} [\, 1 + \frac{1}{z-1} + \frac{1}{(z-1)^2} + \frac{1}{(z-1)^3} + \cdots ....]$$

$$= \frac{1}{(z-1)} + \frac{2}{(z-1)} + \frac{2}{(z-1)^2} + \frac{2}{(z-1)^3} + .....$$

$$= \frac{1}{(z-1)} + \frac{2}{(z-1)^2} + \frac{2}{(z-1)^3} + .....$$

## 2.13 Residue theorem:

If f (z) has a pole at z = a, then |f (z) | → ∞ as z → a.

Method of finding residues:

(a) Residue at simple pole

(i) If f (z) has a simple pole at z = a then

$$\text{Res f (a)} = \lim_{z \to a} (z-a) f(z)$$

(ii) If f (z) is of the form f (z) = $\frac{\varphi(z)}{\psi(z)}$ where $\psi(a)$ = 0 but $\varphi(a) \neq 0$

$$\text{Res (at z = a)} = \frac{\varphi(z)}{\psi'(z)}$$

(b) Residue at a pole of order n.

If f (z) has a pole of order n at z =a then

$$\text{Res (at z = a)} = \frac{1}{(n-1)!} \left\{ \frac{d^{n-1}}{dz^{n-1}}((z-a)^n \, f(z)) \right\}$$

(c) Residue at a pole z = a of any order (simple or of order m)

Res f (a) = coefficient of $\frac{1}{t}$

(d) Residue of f (z) at z = ∞ = $\lim_{z \to \infty} (-z\, f(z))$

The residue of f (z) at infinity = - $\frac{1}{2\pi i} \int_c f(z) dz$

**Example:**

Find the residue at z = 0 of z cos $\frac{1}{z}$

**Solution:**

Expanding the function in powers of $\frac{1}{z}$ $, we\ have$

$$z \cos\frac{1}{z} = z\left[1 - \frac{1}{1!\,2\,z^2} + \frac{1}{4!\ z^4} - \ldots\ldots\ldots\ldots\right]$$

$$= z - \frac{1}{2z} + \frac{1}{24z^3} - \ldots\ldots\ldots\ldots\ldots\ldots$$

The Laurent's expansion about z = 0.

The coefficient of $\frac{1}{z}$ in it is $\frac{-1}{2}$. So the residue of z cos $\frac{1}{z}$ at z = 0 is $\frac{-1}{2}$

## 2.14 PROBABILITY

Probability is a concept which numerically measure the degree of uncertainty and therefore certainty of the occurrence of events. If an event *A* can happen in *m* ways, and fail in *n* ways, all these ways being equally likely to occur, then the probability of the happening of *A* is

= (Number of favorable cases) / (Total number of mutually exclusive and equally likely cases)

$$= \frac{m}{m+n}$$

and that of its failing is defined as $\frac{n}{m+n}$

.

If the probability of the happening = p

$$p + q = C + \frac{n}{m+n}$$

$$= \frac{m+n}{m+n} = 1$$

Or p + q = 1

**DEFINITIONS**

1. **Die :** It is a small cube. Dots are . .. ... :: :.: ::: marked on its faces. Plural of thedie is dice. On throwing a die, the outcome is the number of dots on its upper face.

2. **Cards :** A pack of cards consists of four suits *i.e.* Spades, Hearts, Diamonds and Clubs.Each suit consists of 13 cards, nine cards numbered 2, 3, 4, ..., 10, an Ace, a King, a

Queen and a Jack or Knave. Colour of Spades and Clubs is black and that of Hearts and Diamonds is red. Aces, Kings, Queens, and Jacks are known as *face* cards.

3. **Exhaustive Events or Sample Space :** The set of all possible outcomes of a singleperformance of an experiment is exhaustive events or sample space. Each outcome is called a sample point. In case of tossing a coin once, $S = (H, T)$ is the *sample space.* Two outcomes – Head and Tail – constitute an exhaustive event because no other outcome is possible.

4. **Random Experiment :** There are experiments, in which results may be altogether different, even though they are performed under identical conditions. They are known as random experiments. Tossing a coin or throwing a die is random experiment

5. **Trial and Event :** Performing a random experiment is called a *trial* and outcome is termed as *event.* Tossing of a coin is a trial and the turning up of head or tail is an event.

6. **Equally likely events :** Two events are said to be *'equally likely'*, if one of them cannot be expected in preference to the other. For instance, if we draw a card from well-shuffled pack, we may get any card, then the 52 different cases are equally likely.

7. **Independent events :** Two events may be *independent*, when the actual happening of one does not influence in any way the probability of the happening of the other.

**Example.** The event of getting head on first coin and the event of getting tail on the second coin in a simultaneous throw of two coins are independent.

8. **Mutually Exclusive events :** Two events are known as *mutually exclusive*, when the occurrence of one of them excludes the occurrence of the other. For example, on tossing of a coin, either we get head or tail, but not both.

9. **Compound Event :** When two or more events occur in composition with each other, the simultaneous occurrence is called a compound event. When a die is thrown, getting a 5 or 6 is a compound event.

10. **Favourable Events :** The events, which ensure the required happening, are said to be favourable events. For example, in throwing a die, to have the even numbers, 2, 4 and 6 are favourable cases.

11. **Conditional Probability :** The probability of happening an event $A$, such that event $B$ has already happened, is called the conditional probability of happening of $A$ on the condition that $B$ has already happened. It is usually denoted by $P(A/B)$.

12. **Odds in favour of an event and odds against an event** If number of favourable ways = $m$, number of not favourable events = $n$

($i$) Odds in favour of the event = $m / n$

Odds against the event = $n / m$

13. **Classical Definition of Probability.** If there are $N$ equally likely, mutually, exclusive and exhaustive of events of an experiment and m of these are favourable, then the probability of the happening of the event is defined as $m / N$..

14. **Expected value.** If $p_1, p_2, p_3 \dots p_n$ of the probabilities of the events $x_1, x_2, x_3 \dots x_n$ respectively then expected value

$$E(x) = p_1 x_1 + p_2 x_2 + p_3 x_3 + \dots + p_n x_n = \sum_{\nu=}^{n} p_r x_r$$

## Example:

Find the probability of throwing

(a) 5,

(b) an even number with an ordinary six faced die.

**Solution.**

($a$) There are 6 possible ways in which the die can fall and there is only one way of throwing 5.

$$\text{Probability} = \frac{\textit{Number of favourable ways}}{\text{Total number of equally likely ways}}$$

$$= 1 / 6$$

($b$) Total number of ways of throwing a die = 6

Number of ways falling 2, 4, 6 = 3

The required probability $= \frac{3}{6} = \frac{1}{2}$

**Example:**

Find the probability of throwing 9 with two dice.

**Solution.**

Total number of possible ways of throwing two dice

$= 6 \times 6 = 36.$

Number of ways getting 9

*i.e.* $(3 + 6), (4 + 5), (5 + 4), (6 + 3) = 4$

The required probability = $\frac{4}{36} = \frac{1}{9}$

## Example:

From a pack of 52 cards, one is drawn at random. Find the probability of getting a king.

**Solution.**

A king can be chosen in 4 ways. But a card can be drawn in 52 ways.

$\therefore$ The required probability $= \frac{4}{52} = \frac{1}{13}$

## 2.15 ADDITION LAW OF PROBABILITY

If $p_1, p_2, \ldots\ldots, p_n$ be separate probabilities of mutually exclusive events, then the probability $P$, that any of these events will happen is given by $P = p_1 + p_2 + p_3 + \ldots\ldots + p_n$

**Proof.** Let $A, B, C, \ldots\ldots$ be the events, where probabilities are respectively $p_1, p_2, \ldots\ldots p_n$.

Let $n$ be the total number of favourable cases to either $A$ or $B$ or $C$ or.........

$$= m_1 + m_2 + m_3 + \ldots\ldots + m_n$$

Hence $P\,(A + B + C\ldots) = \dfrac{m_1 + m_2 + m_3 + \ldots\ldots + m_n}{n}$

$$= \frac{m_1}{n} + \frac{m_2}{n} + \frac{m_3}{n} \ldots. + \frac{m_n}{n}$$

$$= P\,(A) + P\,(B) + P\,(C) + \ldots\ldots$$

$$P = p_1 + p_2 + p_3 + \ldots\ldots + p_n \quad \textbf{Proved}$$

## NOT MUTUALLY EXCLUSIVE EVENTS

Consider the case where two events $A$ and $B$ are not mutually exclusive. The probability of the event that either $A$ or $B$ or both occur is given as

$$P\,(A \cup B) = P\,(A) + P\,(B) - P\,(A \cap B)$$

**Example .** An urn contains 10 black and 10 white balls. Find the probability of drawing two balls of the same colour.

**Solution.** Probability of drawing two black balls $= 10_{c_2} \;/\; 20_{c_2}$

$\therefore$ Probability of drawing two balls of the same colour $= \frac{10_{c_2}}{20_{c_2}} + \frac{10_{c_2}}{20_{c_2}}$

$$= 2\,\frac{10_{c_2}}{20_{c_2}}$$

$$= \frac{9}{19}$$

## Example:

A bag contains four white and two black balls and a second bag contains three of each colour. A bag is selected at random, and a ball is then drawn at random from the bag chosen. What is the probability that the ball drawn is white ?

**Solution.**

There are two mutually exclusive cases,

*(i)* when the first bag is chosen,

*(ii)* when the second bag is chosen.

Now the chance of choosing the first bag is $\frac{1}{2}$ and if this bag is chosen, the probability of drawing a white ball is $\frac{4}{6}$ Hence the probability of drawing a white ball from first bag is

$$\frac{1}{2} \times \frac{4}{6} = \frac{1}{3}$$

Similarly the probability of drawing a white ball from second bag is

$$\frac{1}{2} \times \frac{3}{6} = \frac{1}{4}$$

Since the events are mutually exclusive the required probability

$$= \frac{1}{3} + \frac{1}{4}$$

$$= \frac{7}{12}.$$

## 2.16 MULTIPLICATION LAW OF PROBABILITY

If there are two independent events the respective probabilities of which are known, then the probability that both will happen is the product of the probabilities of their happening respectively.

$$cP(A) \times P(B)$$

**Proof.**

Suppose $A$ and $B$ are two independent events. Let $A$ happen in $m1$ ways and fail in $n1$ ways.

$$P(A) = \frac{m_1}{m_1+n_1}$$

Also let $B$ happen in $m_2$ ways and fail in $n_2$ ways.

$$P(B) = \frac{m_2}{m_2+n_2}$$

Now there are four possibilities

$A$ and $B$ both may happen, then the number of ways $= m_1 \,.\, m_2$.

$A$ may happen and $B$ may fail, then the number of ways $= m_1 \,.\, n_2$

$A$ may fail and $B$ may happen, then the number of ways $= n_1 \,.\, m_2$

$A$ and $B$ both may fail, then the number of ways $= n_1 \,.\, n_2$

Thus the total number of ways = $(m_1+n_1)(m_2+n_2)$

Hence the probabilities of the happening of both $A$ and $B$

$$P(AB) = \frac{m_1.m_2}{(m_1+n_1)(m_2+n_2)}$$

$$= \frac{m_1}{m_1+n_1} \cdot \frac{m_2}{m_2+n_2}$$

$$= P(A).P(B)$$

## Example:

The probability that machine A will be performing an usual function in 5 years' time is $\frac{1}{4}$ , while the probability that machine B will still be operating usefully at the end of the same period, is $\frac{1}{3}$. Find the probability in the following cases that in 5 years time:

(i) Both machines will be performing an usual function.

(ii) Neither will be operating.

(iii) Only machine B will be operating.

(iv) At least one of the machines will be operating.

**Solution.**

$P$ ($A$ operating usefully) $= \frac{1}{4}$ , q (A) $= 1-\frac{1}{4} = \frac{3}{4}$

$P$ ($B$ operating usefully) $= \frac{1}{3}$, q (B) $= 1 - \frac{1}{3} = \frac{2}{3}$

*(i)* $P$ (Both $A$ and $B$ will operate usefully) $= P(A) \cdot P(B)$

$$= \frac{1}{4} \times \frac{1}{3} = \frac{1}{12}$$

*(ii)* $P$ (Neither will be operating) $= q(A) \cdot q(B)$

$$= \frac{3}{4} \times \frac{2}{3} = \frac{1}{2}$$

*(iii)* $P$ (Only $B$ will be operating) $= p(B) \times q(A)$

$$= \frac{1}{3} \times \frac{3}{4} = \frac{1}{4}$$

*(iv)* $P$ (At least one of the machines will be operating)

$$= 1 - P = 1 - P$$

$$=1 - \frac{1}{2} = \frac{1}{2}$$

## Example:

There are two groups of subjects one of which consists of 5 science and 3 engineering subjects and the other consists of 3 science and 5 engineering subjects. An unbiased die is cast. If number 3 or number 5 turns up, a subject is selected at random from the first group, otherwise the subject is selected at random from the second group. Find the probability that an engineering subject is selected ultimately.

**Solution:**

Probability of turning up 3 or 5 = $\frac{2}{6} = \frac{1}{3}$

Probability of selecting engineering subject from first group = $\frac{3}{8}$

Now the probability of selecting engineering subject from first group on turning up 3 or 5

$$= \frac{1}{3} \text{ x } \frac{3}{8} = \frac{1}{8}$$

Probability of not turning up 3 or 5 $= 1 - \frac{1}{3}$

$$= \frac{2}{3}$$

Probability of selecting engineering subject from second group = $\frac{5}{8}$

$$= \frac{2}{3} \text{ x } \frac{5}{8}$$

$$= \frac{5}{12}$$

Probability of the selection of engineering subject $= \frac{1}{8} + \frac{5}{12}$

$$= \frac{13}{24}$$

## 2.17 Introduction of statistics:

**Statistics** is a branch of science dealing with the collection of data, organising, summarising, presenting and analysing data and drawing valid conclusions and thereafter making reasonable decisions on the basis of such analysis.

**Frequency distribution** is the arranged data, summarised by distributing it into classes or categories with their frequencies.

**Wages of 100 workers**

| | | | | | |
|---|---|---|---|---|---|
| Wages in Rs. | 0-10 | 10-20 | 20-30 | 30-40 | 40-50 |
| Numbers of workers | 12 | 23 | 35 | 20 | 10 |

**Graphical representation.** It is often useful to represent frequency distribution by means of a diagram. The different types of diagrams are

1. Histogram
2. Frequency polygon
3. Frequency curve
4. Cumulative frequency curve or give
5. Bar chart
6. Circles or Pie diagrams.

## 2.18 Average or measures of central tendency:

An average is a value which is representative of a set of data. Average value may also be termed as measures of central tendency. There are five types of averages in common.

(i) Arithmetic average or mean

(ii) Median

(iii) Mode

(iv) Geometric Mean

(v) Harmonic Mean

## Arithmetic mean:

If $x_1, x_2, x_3, \ldots\ldots x_n$ are $n$ numbers, then their arithmetic mean (A.M.) is defined by

$$A.M. = \frac{x_1 + x_2 + x_3 + \cdots \ldots\ldots + x_n}{n} = \frac{\sum x}{n}$$

If the number $x1$ occurs $f_1$ times, $x_2$ occurs $f_2$ times and so on, then

$$A.M. = \frac{f_1 x_1 + f_2\, x_2 + f_3\, x_3 + \cdots \ldots\ldots + \ f_n\, x_n}{f_1 + f_2 + f_3 + \cdots \ldots.. + f_n} = \frac{\sum f x}{\Sigma f}$$

This is known as direct method.

**Example:**

*Find the mean of 20, 22, 25, 28, 30.*

**Solution:**

$$A.M. = \frac{20+22+25+28+30}{5}$$

$$= \frac{125}{5}$$

$$= 25$$

## Example .

Find the mean of the following :

| Numbers | 8 | 10 | 15 | 20 |
|---|---|---|---|---|
| frequncy | 5 | 8 | 8 | 4 |

## Solution:

$$\Sigma f x = 8 \times 5 + 10 \times 8 + 15 \times 8 + 20 \times 4$$

$$= 40 + 80 + 120 + 80$$

$$= 320$$

$$\Sigma f = 5 + 8 + 8 + 4 = 25$$

$$A.M. = \frac{\sum fx}{\Sigma f}$$

$$= \frac{320}{25}$$

$$\boldsymbol{A.M.} = 12.8$$

**Short cut method**

Let $a$ be the assumed mean, $d$ the deviation of the variate $x$ from $a$. Then

$$\frac{\sum fd}{\Sigma f} = \frac{\sum f(x-a)}{\Sigma f}$$

$$= \frac{\sum fx}{\Sigma f} - \frac{\sum fa}{\Sigma f}$$

$$\frac{\sum fd}{\Sigma f} = \text{A.M} - \frac{a\sum f}{\Sigma f}$$

$$= \text{A.M} - a$$

$$\text{A.M} = a + \frac{\sum fd}{\Sigma f}$$

**Example**

Find the arithmetic mean for the following distribution

| class | 0-10 | *10 – 20* | *20 - 30* | *30 - 40* | *40 - 50* |
|---|---|---|---|---|---|
| frequency | *7* | *8* | *20* | *10* | *5* |

**Solution:**

Let assumed mean ($a$) = 25.

| Class | Mid value x | *Frequency Y* | *x-25 = d* | *F . d* |
|---|---|---|---|---|
| 0 – 10 | *5* | *7* | *-20* | -140 |
| 10 – 20 | *15* | *8* | -10 | -80 |
| 20 – 30 | 25 | *20* | 0 | 0 |
| 30 – 40 | 35 | *10* | +10 | +100 |
| 40 – 50 | 45 | *5* | +20 | +100 |
| Total | | *50* | | -20 |

$$\text{A.M} = a + \frac{\sum fd}{\Sigma f}$$

$$= 25 + \frac{-20}{50}$$

$$= 24.6.$$

## <u>Step deviation method</u>

Let $a$ be the assumed mean, $i$ the width of the class interval and

$$D = \frac{x-a}{i},$$

$$\text{A.M} = a + \frac{\sum fD}{\Sigma f}\, i$$

## Example:

Find the arithmetic mean of the data given above example 3 by step deviation method.

**Solution.**

$a = 25$

| **Class** | **Mid value x** | ***Frequency Y*** | $D = \frac{x-a}{i}$ | ***f. D*** |
|---|---|---|---|---|
| 0 – 10 | *5* | *7* | *-2* | -14 |
| 10 – 20 | *15* | *8* | -1 | -8 |
| 20 – 30 | 25 | *20* | 0 | 0 |
| 30 – 40 | 35 | *10* | +1 | +10 |
| 40 – 50 | 45 | *5* | + | +10 |
| Total | | *50* | | -2 |

$$\mathbf{A}.\text{M} = a + \frac{\sum fD}{\Sigma f} \cdot i$$

$$= 25 + \frac{-25}{50} \times 10$$

$$\text{A.M} = 24.6$$

## 2.19 Median:

Median is defined as the measure of the central item when they are arranged in ascending or descending order of magnitude

. When the total number of the items is odd and equal to say *n*, then the value of $\frac{1}{2}(n+1)$th item gives the median.

When the total number of the frequencies is even, say *n*, then there are two middle items, and so the mean of the values of $\frac{1}{2}{}^{\text{th}}$ and $\frac{1}{2}(n+1)^{\text{th}}$ items in the median.

**Example:**

Find the median of 6, 8, 9, 10, 11, 12, 13.

**Solution.**

Total number of items = 7

The middle item = $\frac{1}{2}\,(7+1)^{\text{th}} = 4^{\text{th}}$

Median = Value of the 4th item = 10

For grouped data, Median = $l + \dfrac{\left(\frac{1}{2}\right)(N-F)}{f}\,i$

where $l$ is the lower limit of the median class, $f$ is the frequency of the class, $i$ is the width of the class-interval, $F$ is the total of all the preceding frequencies of the median-class and $N$ is total frequency of the data.

**Example**

Find the value of Median from the following data

| **No. of days for which absent (less than)** | 5 | *10* | 15 | *20* | 25 | *30* | *35* | *40* | *45* |
|---|---|---|---|---|---|---|---|---|---|
| **No. of students** | 29 | 224 | 465 | 582 | 634 | 644 | 650 | 653 | 655 |

**Solution:**

The given cumulative frequency distribution will first be converted into ordinary frequency as under

| **Class – Interval** | **Cumulative frequency** | **Ordinary frequency** |
|---|---|---|
| 0 – 5 | *29* | 29 = 29 |
| 5 -10 | 224 | 224 – 29 = 195 |
| 10 – 15 | 465 | 465 – 224 = 241 |
| 15 – 20 | 582 | 582 – 465 = 117 |
| 20 – 25 | 634 | 634 – 582 = 52 |
| 25 – 30 | 644 | 644 – 634 = 10 |
| 30 – 35 | 650 | 650 – 644 = 6 |
| 35 – 40 | 653 | 653 – 650 = 3 |
| 40 – 45 | 655 | 655 – 653 = 2 |

Median = size of $\frac{655}{2}$ or $327.5^{\text{th}}$ item

$327.5^{th}$ item lies in 10–15 which is the median class.

$$M = l + \frac{\left(\frac{N}{2}\right) - C}{f}\, i$$

Where *l* stands for lower limit of median class,

*N* stands for the total frequency,

*C* stands for the cumulative frequency just preceding the median class,

*i* stands for class interval

*f* stands for frequency for the median class.

$$\text{Median} = 10 + \frac{\left(\frac{655}{2}\right) - 224}{241} \times 5$$

$$= 10 + 2.15$$

**Median = 12.15.**

## 2.20 MODE

Mode is defined to be the size of the variable which occurs most frequently.

**Example**

Find the mode of the following items :

0, 1, 6, 7, 2, 3, 7, 6, 6, 2, 6, 0, 5, 6, 0.

**Solution.**

6 occurs 5 times and no other item occurs 5 or more than 5 times, hence the mode is 6.

For grouped data,

$$\textbf{Mode} = l + \frac{f - f_{-1}}{2f - f_{-1} - f_1} \cdot i$$

where *l* is the lower limit of the modal class, *f* is the frequency of the modal class, *i* is the width of the class, $f_{-1}$ is the frequency before the modal class and *f*1 is the frequency after the modal class.

## 2.21 STANDARD DEVIATION

Standard deviation is defined as the square root of the mean of the square of the deviation from the arithmetic mean.

$$S.D. = \sigma = \frac{\sqrt{\sum f(x-\bar{x})^2}}{\sum f}$$

**Note.** 1. The square of the standard deviation $\sigma^2$ is called variance.

2. $\sigma^2$ is called the second moment about the mean and is denoted by $\mu_2$.

**SHORTEST METHOD FOR CALCULATING STANDARD DEVIATION**

We know that $\sigma^2 = \frac{1}{N} \sum f(x - \bar{x})^2$

$$= \frac{1}{N} \sum f(x - a - \overline{x - a})^2$$

$$= \frac{1}{N} \sum f(d - \overline{x - a})^2$$

where $x - a = d$

$$= \frac{1}{N} \sum f\, d^2 - 2(-a + \bar{x})^2 \frac{1}{N} \sum fd(-a + \bar{x})^2 \frac{1}{N} \sum f \sum f$$

$$= \frac{1}{N} \sum f\, d^2 - 2(-a + \bar{x})^2 \frac{1}{N} \sum f\, d^1 + 2(-a + \bar{x})^2$$

$$\bar{x} = a + \frac{\sum fd}{N} \quad \text{or} \quad \bar{x} - a = \frac{\sum fd}{N}$$

$$\sigma^2 = \frac{1}{N} \sum f\, d^2 - 2 \frac{\sum fd}{N} \frac{1}{N} \sum fd + \left(\sum fd/_N\right)^2$$

$$= \frac{1}{N} \sum f\, d^2 \left(\sum fd/_N\right)^2$$

$$\textbf{S.D} = \sqrt{\frac{\sum fd^2}{N} - \left(\frac{\Sigma fd}{N}\right)^2}$$

## Example:

Calculate the mean and standard deviation for the following data :

| Size of item | 6 | *17* | *8* | *9* | 10 | *11* | *12* |
|---|---|---|---|---|---|---|---|
| Frequency | 3 | 6 | 9 | 13 | 8 | 5 | 4 |

**Solution:**

Assumed mean = 9

| x | F | D = x- a | f.d | *f. d* $^2$ |
|---|---|---|---|---|
| 6 | *3* | -3 | -9 | 27 |
| 7 | 6 | -2 | -12 | 24 |
| 8 | 9 | -1 | -9 | 9 |
| 9 | 13 | 0 | 0 | 0 |
| 10 | 8 | +1 | 8 | 8 |
| 11 | 5 | +2 | 10 | 20 |
| 12 | 4 | +3 | 12 | 36 |
| | $\sum f = 48$ | | $\sum f\,d = 0$ | $\sum f\,d^2 = 124$ |

$$\text{Mean} = a + \frac{\sum fd}{\Sigma f}$$

$$= 9 + 0$$

$$= 9$$

$$S.D. = \frac{\sqrt{\sum f(x-\bar{x})^2}}{\sum f}$$

$$= \sqrt{\frac{\sum fd^2}{N} - \left(\frac{\Sigma fd}{N}\right)^2}$$

$$= \frac{\sqrt{124}}{48}$$

$$\boldsymbol{S.D} = 1.6$$

## Example:

From the following frequency distribution, compute the standard deviation of 100 students *:*

| Mass in kg | 60 - 62 | 63 – 65 | 66 - 68 | 69 - 71 | *72* - 74 |
|---|---|---|---|---|---|
| Number of students | 5 | *18* | *42* | *27* | *8* |

## Solution:

Assumed mean = 67

| Mass in kg | No. of students f | x | d = x - 67 | f.d | *f. d* $^2$ |
|---|---|---|---|---|---|
| 60 - 62 | 5 | 61 | -6 | -30 | 180 |
| 63 – 65 | 18 | 64 | -3 | -54 | 162 |
| 66 – 68 | 42 | 67 | 0 | 0 | 0 |
| 69 – 71 | 27 | 70 | 3 | 81 | 243 |
| 72 - 74 | 8 | 73 | 6 | 48 | 288 |
| | $\sum f = 100$ | | | $\sum f\,d = 45$ | $\sum f\,d^2 = 124$ |

$$\text{S.D} = \sqrt{\frac{\Sigma f\ d^2}{\sum f} - \left(\frac{\Sigma f d}{\Sigma d}\right)^2}$$

$$= \sqrt{\frac{873}{100} - \left(\frac{45}{100}\right)^2}$$

$$= \sqrt{8.73 - 0.2025}$$

$$= \sqrt{8.5275}$$

**S.D = 2.9202**.

## Example:

Compute the standard deviation for the following frequency distribution

| **Class interval** | 0 - 4 | 4 - 8 | 8 - 12 | 12 - 16 |
|---|---|---|---|---|
| **Frequency** | 4 | 8 | 2 | 1 |

## Solution:

Assumed mean = 6

| **Class interval** | **f** | **X** | **d = x - 6** | **f.d** | ***f. d²*** |
|---|---|---|---|---|---|
| 0 – 4 | 4 | 2 | -4 | -16 | 64 |
| 4 – 8 | 8 | 6 | 0 | 0 | 0 |
| 8 – 12 | 2 | 10 | +4 | 8 | 32 |
| 12 - 16 | 1 | 14 | +8 | 8 | 64 |
| | $\sum f = 15$ | | | $\sum f\,d = 0$ | $\sum f\,d^2 = 160$ |

$$\text{S.D} = \sqrt{\frac{\Sigma f\ d^2}{\Sigma f} - \left(\frac{\Sigma f d}{\Sigma d}\right)^2}$$

$$= \sqrt{\frac{160}{15} - (0)^2}$$

**S.D = 3.266**

**UNIT III: MATRICES**

Types of Matrices and their properties, Rank of a Matrix-Conjugate of a matrix - Adjoint of a matrix - Inverse of a matrix - Hermitian and Unitary Matrices - Trace of a matrix-Transformation of matrices - Characteristic equation - Eigen values and Eigen vectors - Cayley–Hamilton theorem –Diagonalization

## 3.1 Types of Matrices and their properties

Let us consider a set of simultaneous equations,

$$x + 2y + 3z + 5t = 0$$

$$4x + 2y + 5z + 7t = 0$$

$$3x + 4y + 2z + 6t = 0.$$

$$5x + 7y + 9z + 0t = 0.$$

Now we write down the coefficients of *x, y, z, t* of the above equations and enclose them within brackets and then we get

$$A = \begin{bmatrix} 1 & 2 & 3 & 5 \\ 4 & 2 & 5 & 7 \\ 3 & 4 & 2 & 6 \\ 5 & 7 & 9 & 8 \end{bmatrix}$$

The above system of numbers, arranged in a rectangular array in rows and columns and bounded by the brackets, is called a matrix.

It has got 3 rows and 4 columns and in all 3 × 4 = 12 elements. It is termed as 3 × 4 matrix, to be read as [3 by 4 matrix]. In the double subscripts of an element, the first subscript determines the row and the second subscript determines the column in which the element lies, *aij* lies in the *i*th row and *j*th column.

## 3.2 VARIOUS TYPES OF MATRICES

(*a*) **Row Matrix.** If a matrix has only one row and any number of columns, it is called a *Row matrix*, *e.g.*,[2 7 3 9] (*b*) **Column Matrix.** A matrix, having one column and any number of rows, is called a *Column matrix, e.g.,* $\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$

(*c*) **Null Matrix or Zero Matrix.** Any matrix, in which all the elements are zeros, is called a *Zero matrix* or *Null matrix e.g.,*

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

(*d*) **Square Matrix.** A matrix, in which the number of rows is equal to the number of columns, is called a square matrix *e.g.,*

$$\begin{bmatrix} 1 & 7 \\ 7 & 4 \end{bmatrix}$$

(*e*) **Diagonal Matrix.** A square matrix is called a diagonal matrix, if all its non-diagonal elements are zero *e.g.,*

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 8 \end{bmatrix}$$

(*f* ) **Scalar matrix.** A diagonal matrix in which all the diagonal elements are equal to a scalar, say (*k*) is called a scalar matrix.

*For example* $\begin{bmatrix} -6 & 0 & 0 & 0 \\ 0 & -6 & 0 & 0 \\ 0 & 0 & -6 & 0 \\ 0 & 0 & 0 & -6 \end{bmatrix}$

(*g*) **Unit or Identity Matrix.** A square matrix is called a unit matrix if all the diagonal elements are unity and non-diagonal elements are zero *e.g.,*

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(*h*) **Symmetric Matrix.** A square matrix will be called symmetric, if for all values of $i$ and $j$, $a_{ij} = a_{ji}$ *i.e.,* $A' = A$

$$\begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}$$

(*i*) **Skew Symmetric Matrix.** A square matrix is called skew symmetric matrix, if

(1) $a_{ij} = -a_{ji}$ for all values of $i$ and $j$, or $A' = -A$

(2) All diagonal elements are zero, *e.g.*,

$$\begin{bmatrix} 0 & -h & -g \\ h & 0 & -f \\ g & f & 0 \end{bmatrix}$$

(*j*) **Triangular Matrix.** (Echelon form) A square matrix, all of whose elements below the leading diagonal are zero, is called an *upper triangular matrix.* A square matrix, all of whose elements above the leading diagonal are zero, is called *a lower triangular matrix*

*e.g.*, $\begin{bmatrix} 1 & 7 & 9 \\ 0 & 5 & 2 \\ 0 & 0 & 6 \end{bmatrix}$ $\begin{bmatrix} 3 & 0 & 0 \\ 8 & 5 & 0 \\ 9 & 1 & 7 \end{bmatrix}$

Upper triangular matrix lower triangular matrix

(*k*) **Transpose of a Matrix.** If in a given matrix $A$, we interchange the rows and the corresponding columns, the new matrix obtained is called the transpose of the matrix $A$ and is denoted by $A'$ or $A^T$

*e.g.*, $A = \begin{bmatrix} 2 & 3 & 4 \\ 1 & 0 & 5 \\ 6 & 7 & 8 \end{bmatrix}$ $A' = \begin{bmatrix} 2 & 1 & 6 \\ 3 & 0 & 7 \\ 4 & 5 & 8 \end{bmatrix}$

(*l*) **Orthogonal Matrix.** A square matrix $A$ is called an orthogonal matrix if the product of the matrix $A$ and the transpose matrix $A'$ is an identity matrix *e.g.*,

$$A.\, A' = I$$

If $|A| = 1$, matrix $A$ is proper.

(*m*) **Conjugate of a Matrix**

$$A = \begin{bmatrix} 1+i & 2-3i \\ 7+2i & -i \end{bmatrix}$$

Conjugate of matrix $A$ is $\bar{A}$

$$\bar{A} = \begin{bmatrix} 1-i & 2+3i \\ 7-2i & +i \end{bmatrix}$$

(*n*) **Matrix $A^\theta$.** Transpose of the conjugate of a matrix $A$ is denoted by $A^\theta$

$$A = \begin{bmatrix} 1+i & 2-3i & 4 \\ 7+2i & -i & 3-2i \end{bmatrix}$$

$$\bar{A} = \begin{bmatrix} 1-i & 2+3i & 4 \\ 7+2i & +i & 3+2i \end{bmatrix}$$

$$(\bar{A})' = \begin{bmatrix} 1-i & 7-2i \\ 2+3i & i \\ 4 & 3+2i \end{bmatrix}$$

$$\mathbf{A}^{\theta} = \begin{bmatrix} 1-i & 7-2i \\ 2+3i & i \\ 4 & 3+2i \end{bmatrix}$$

(*o*) **Unitary Matrix.** A square matrix $A$ is said to be unitary if

$$A^{\theta} A = I$$

$$A = \begin{bmatrix} \frac{1+i}{2} & \frac{-1+i}{2} \\ \frac{1+i}{2} & \frac{1-i}{2} \end{bmatrix}$$

$$A^{\theta} = \begin{bmatrix} \frac{1-i}{2} & \frac{+1-i}{2} \\ \frac{-1-i}{2} & \frac{-1+i}{2} \end{bmatrix}$$

$$A A^{\theta} = I$$

(*p*) **Hermitian Matrix.** A square matrix $A = (a_{ij})$ is called Hermitian matrix, if every $i$-$j^{th}$ element of $A$ is equal to conjugate complex $j - i$ th element of $A$.

In other words, $a_{ij} = \overline{a_{ij}}$

$$\begin{bmatrix} 1 & 2+3i & 3+i \\ 2-3i & 2 & 1-2i \\ 3-i & 1+2i & 5 \end{bmatrix}$$

Necessary and sufficient condition for a matrix $A$ to be Hermitian is that $A = A^{\theta}$ *i.e.* conjugate transpose of $A$

$$\Rightarrow A = (A)'.$$

(*q*) **Skew Hermitian Matrix.** A square matrix $A = (a_{ij})$ will be called a Skew Hermitian matrix if every $i$-$j$th element of $A$ is equal to negative conjugate complex of $j$-$i^{th}$ element of $A$.

In other words, $a_{ij} = -\ \overline{a_{ij}}$

All the elements in the principal diagonal will be of the form

$$a_{ii} = -\ \overline{a_{ii}}$$

or $a_{ii} + \overline{a_{ii}} = 0$

If $a_{ii} = a + ib$

then $\overline{a_{ii}} = a - ib$

$(a + ib) + (a - ib) = 0$

$2a = 0$

$a = 0$

So, *aii* is pure imaginary

$a_{ii} = 0.$

Hence, all the diagonal elements of a Skew Hermitian Matrix are either zeros or pure imaginary.

$$\begin{bmatrix} i & 2-3i & 4+5i \\ -(2+3i) & 0 & 2i \\ -(4-5i) & 2i & -3i \end{bmatrix}$$

The necessary and sufficient condition for a matrix $A$ to be Skew Hermitian is that

$$A^{\theta} = -A$$

$$(A)' = -A$$

($r$) **Idempotent Matrix.** A matrix, such that $A2 = A$ is called Idempotent Matrix.

$$e.g.\ A = \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix} \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix} = A$$

($s$) **Periodic Matrix.** A matrix $A$ will be called a Periodic Matrix, if

$$A^{k+1} = A$$

where $k$ is a +ve integer. If $k$ is the least + ve integer, for which $Ak+1 = A$, then $k$ is said to be the period of $A$. If we choose $k = 1$, we get $A^2 = A$ and we call it to be idempotent matrix.

($t$) **Nilpotent Matrix.** A matrix will be called a Nilpotent matrix, if $A^k$ = 0 (null matrix) where $k$ is a +ve integer ; if however $k$ is the least +ve integer for which $^kk = 0$, then $k$ is the *index* of the nilpotent matrix.

e.g = A = $\begin{bmatrix} ab & b^2 \\ -a^2 & -ab \end{bmatrix}$,

$$A^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, = 0$$

$A$ is nilpotent matrix whose index is 2.

($u$) **Involuntary Matrix.** A matrix $A$ will be called an Involuntary matrix, if $A^2 = I$ (unit matrix). Since $I_2 = I$ always $\therefore$ Unit matrix is involuntary.

($v$) **Equal Matrices.** Two matrices are said to be equal if

($i$) They are of the same order.

($ii$) The elements in the corresponding positions are equal.

$$A = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}$$

$$B = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}$$

$$A = B$$

($w$) **Singular Matrix.** If the determinant of the matrix is zero, then the matrix is known as singular matrix

e.g. $A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$

is singular matrix, because $|A| = 6 – 6 = 0$.

## 3.3 Adjoint of matrix:

Let the determinant of the square matrix $A$ be $| A |$.

If $A = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$ $\quad |A| = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$

The matrix formed by the co-factors of the elements in

$| A |$ is $\begin{bmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{bmatrix}$

$$A1 = \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} = b_2c_3 – b_3c_2$$

$$A2= \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} = b_1c_3 - b_3c_1$$

$$A3 = \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix} = b_1c_2 - b_2c_1$$

$$B1 = \begin{vmatrix} a_2 & a_3 \\ c_2 & c_3 \end{vmatrix} = a_2c_3 - a_3c_2$$

$$B2 = \begin{vmatrix} a_1 & a_3 \\ c_1 & c_3 \end{vmatrix} = a_1c_3 - a_3c_1$$

$$B3 = \begin{vmatrix} a_1 & a_2 \\ c_1 & c_2 \end{vmatrix} = a_1c_2 - a_2c_1$$

$$C1 = \begin{vmatrix} a_1 & a_2 \\ b_2 & b_1 \end{vmatrix} = a_1b_1 - a_2b_2$$

$$C2 = \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} = a_1b_3 - a_3b_1$$

$$C3 = \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = a_1b_2 - a_2b_1$$

Then the transpose of the matrix of co-factors

$$\begin{bmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{bmatrix}$$

is called the adjoint of the matrix *A* and is written as adj A.

## 3.4 INVERSE OF A MATRIX

If *A* and B are two square matrices of the same order, such that

$$AB = BA = I \ (I = \text{unit matrix})$$

then *B* is called the inverse of *A i.e.* $B = A^{-1}$ and *A* is the inverse of *B.*

*Condition for a square matrix A to possess an inverse is that matrix A is non-singular,*

$$i.e., |A| \neq 0$$

If *A* is a square matrix and *B* be its inverse, then $AB = I$

Taking determinant of both sides, we get $|AB| = |I|$ or

$$|A||B| = I$$

From this relation it is clear that $|A| \neq 0$

*i.e.* the matrix *A* is non-singular.

**To find the inverse matrix with the help of adjoint matrix**

We know that $A \,.\, (Adj.\, A) = |\, A \,|\, I$

$$A \,.\, \frac{1}{|A|}\,(Adj.\, A) = I$$

$$A \,.\, A^{-1} = I$$

$$\therefore\ \mathbf{A^{-1}} = \frac{\mathbf{1}}{|\boldsymbol{A}|}\,\boldsymbol{(Adj.\, A)}$$

## Example

$$If\ A = \begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 4 \end{bmatrix} \qquad \text{find}\ \ A^{-1}$$

## Solution:

$$A = \begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 4 \end{bmatrix}$$

$$|\, A \,| = 3\,(-3+4) + 3\,(2-0) + 4\,(-2-0) = 3 + 6 - 8 = 1$$

The co-factors of elements of various rows of $|\, A \,|$ are

$$\begin{bmatrix} -3+4 & -2-0 & -2 \\ 3-4 & 3-0 & 3 \\ -12+12 & -12+8 & -9+6 \end{bmatrix}$$

Therefore, the matrix formed by the co-factors of

$$|\, A \,| = \begin{bmatrix} 1 & -2 & -2 \\ -1 & 3 & 3 \\ 0 & -4 & -3 \end{bmatrix}$$

$$Adj.\, A = \begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & -4 \\ -2 & 3 & -3 \end{bmatrix}$$

$$\mathbf{A^{-1}} = \frac{\mathbf{1}}{|\boldsymbol{A}|}\,\boldsymbol{(Adj.\, A)}$$

$$= 1\begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & -4 \\ -2 & 3 & -3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & -4 \\ -2 & 3 & -3 \end{bmatrix}$$

**Example:**

*If* $A = \frac{1}{9}\begin{bmatrix} -8 & 1 & 4 \\ 4 & 4 & 7 \\ 1 & -8 & 4 \end{bmatrix}$ , *p*rove that A–1 = A′, A′ being the transpose of A.

**Solution:**

$$A = \frac{1}{9}\begin{bmatrix} -8 & 1 & 4 \\ 4 & 4 & 7 \\ 1 & -8 & 4 \end{bmatrix}$$

$$A' = \frac{1}{9}\begin{bmatrix} -8 & 4 & 1 \\ 1 & 4 & -8 \\ 4 & 7 & 4 \end{bmatrix}$$

$$A\,A' = \frac{1}{9}\begin{bmatrix} -8 & 4 & 1 \\ 1 & 4 & -8 \\ 4 & 7 & 4 \end{bmatrix} \frac{1}{9}\begin{bmatrix} -8 & 1 & 4 \\ 4 & 4 & 7 \\ 1 & -8 & 4 \end{bmatrix}$$

$$= \frac{1}{81}\begin{bmatrix} 64+1+16 & -32+4+28 & -8-8+16 \\ -32+4+28 & 16+16+49 & 4-32+28 \\ -8-8+16 & 4-32+28 & 1+64+16 \end{bmatrix}$$

$$= \frac{1}{81}\begin{bmatrix} 81 & 0 & 0 \\ 0 & 81 & 0 \\ 0 & 0 & 81 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{or}$$

$$A\,A' = \mathrm{I}$$

## 3. 5 RANK OF A MATRIX

The rank of a matrix is said to be *r* if

(*a*) It has at least one non-zero minor of order *r*.

(*b*) Every minor of *A* of order higher than *r* is zero.

**Note:** (*i*) Non-zero row is that row in which all the elements are not zero.

(*ii*) The rank of the product matrix *AB* of two matrices *A* and *B* is less than the rank of either of the matrices *A* and *B*.

## Example

Reduce to normal form the following matrix A = $\begin{bmatrix} 1 & 2 & -1 & 3 \\ 4 & 1 & 2 & 1 \\ 3 & -1 & 1 & 2 \\ 1 & 2 & 0 & 1 \end{bmatrix}$

## Solution:

$$\begin{bmatrix} 1 & 2 & -1 & 3 \\ 4 & 1 & 2 & 1 \\ 3 & -1 & 1 & 2 \\ 1 & 2 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & 7 & 6 & -11 \\ 0 & -7 & 4 & -7 \\ 0 & 0 & 1 & -2 \end{bmatrix}$$

$R_2 \rightarrow R_2 - 4 R_1$

$R_3 \rightarrow R_3 - 3 R_1$

$R_4 \rightarrow R_4 - R_1$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -7 & 6 & -11 \\ 0 & -7 & 4 & -7 \\ 0 & 0 & 1 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -7 & 6 & -11 \\ 0 & 0 & -2 & 4 \\ 0 & 0 & 1 & -2 \end{bmatrix}$$

$R_3 \rightarrow R_3 - R_2$

$C_2 \rightarrow C_2 - 2 C_1$

$C_3 \rightarrow C_3 + C_1$

$C_4 \rightarrow C_4 - 3C_1$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -7 & 0 & 0 \\ 0 & 0 & -2 & 4 \\ 0 & 0 & 1 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -7 & 0 & 0 \\ 0 & 0 & -2 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$R_4 \rightarrow R_4 + 1/2\, R_3$

$C_3 \rightarrow C_3 + 6/7\, C_2$

$C_4 \rightarrow C_4 - 11/7\, C_2$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -7 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$R_2 \rightarrow -1/7\, R_2$

$R_3 \rightarrow -1/2\, R_3$

$C_4 \rightarrow C_4 + 2\, C_3$

Rank of $A = 3$

## 3.6 Eigen Values and Eigen Vectors:

For a square matrix $A$ of order $n$, the number $\lambda$ is an eigenvalue if and only if there exists a non-zero vector $X$ such that

$$A\,X = \lambda\,X$$

Using the matrix multiplication properties, we obtain $(A - \lambda\, I_n)\, X = 0$ This is a linear system for which the matrix coefficient is $A -$ . We also know that this system has one solution if and only if the matrix coefficient is invertible, i.e. $(A - \lambda\, I_n) \neq 0$. Since the zero-vector is a solution and $X$ is not the zero vector, then we have $(A - \lambda\, I_n) = 0$. In general, for a square matrix $A$ of order $n$, the equation

$$(A - \lambda\, I_n) = 0$$

$ie.,$ $$|A - \lambda\, I_n| = 0$$

Will give the eigenvalues of $A$. This equation is called the characteristic equation or characteristic polynomial of $A$. It is a polynomial function of degree $n$. So we know that this equation will not have more than n roots or solutions. So a square matrix $A$ of order n will not have more than n eigenvalues.

## Example:

Find the eigen values eigen vector of the matrix $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$

**Solution:**

Let $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$ and the characteristic equation is $|\, A - \lambda\, I \,| = 0$

$$|\, A - \lambda\, I \,| = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1-\lambda & 0 & 0 \\ 0 & 1-\lambda & 1 \\ 0 & 1 & 1-\lambda \end{bmatrix} = 0$$

$(1 - \lambda) \{ (1 - \lambda)^2 - 1\} = 0$

$(1 - \lambda) \{1 + \lambda^2 - 2\lambda - 1\} = 0$

$(1 - \lambda)(\lambda - 2) \lambda = 0$

ie., $\lambda = 0, 1, 2$

The eigen values of the matrix A are 0, 1, 2

And the eigen value equation is $(A - \lambda\, I)\, X = 0$

**Case 1**

$\lambda = 0$, the eigen value equation is $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

we get $x_1 = 0$

$x_2 + x_3 = 0$

$x_2 + x_3 = 0$

Solving these equations we get $x_1 = 0$ ; $x_2 = -x_3$

$$X_1 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ k \\ -k \end{bmatrix}$$

To normalize the eigen vector it must be equated to unity $|X_1| = 1$,

$$\sqrt{0^2 + k^2 + (-k)^2} = 1$$

$$\sqrt{2k^2} = 1$$

$$K = \frac{1}{\sqrt{2}}$$

$\therefore$ the normalized eigen vector of matrix $A$ for $\lambda = 0$ is $\{0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\}$

**Case 2**

$\lambda = 1$, the eigen value equation is $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

We get $x_3 = 0$

$x_2 = 0$

So that $X_2 = \{x_1, x_2, x_3\} = \{1, 0, 0\}$ is the suitable eigen vector and is normalized.

**Case 3**

$\lambda = 2$, the eigen value equation is $\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

We get $-x_1 = 0$

$-x_2 + x_3 = 0$

$x_2 - x_3 = 0$

Solving these equations we get $x_1 = 0$; $x_2 = x_3$ $X_3 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ k \\ k \end{bmatrix}$

the normalized the eigen vector is

$$\sqrt{0^2 + k^2 + k^2} = 1$$

$$\sqrt{2k^2} = 1$$

$$K = \frac{1}{\sqrt{2}}$$

$\therefore$ the normalized eigen vector of matrix $A$ for $\lambda = 0$ is $\{0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\}$

**3.7 Cayley – Hamilton Theorem**:

Every square matrix satisfies its own characteristics equation. For a square matrix A of order n, the characteristic polynomial is

$$|A - \lambda I| \quad = \quad a_0 + a_1\lambda + a_2\lambda^2 + \ldots\ldots\ldots\ldots + a_n\lambda^n$$

Then the matrix equation $a_0 I + a_1 X + a_2 X^2 + \ldots\ldots\ldots\ldots + a_n X^n = 0$ is satisfied by $X = A$.

**Proof:**

The characteristic polynomial is

$$|A - \lambda I| \quad = \quad a_0 + a_1\lambda + a_2\lambda^2 + \ldots\ldots\ldots\ldots + a_n\lambda^n$$

The characteristic equation of A is

$$|A - \lambda I| \quad = \quad a_0 + a_1\lambda + a_2\lambda^2 + \ldots\ldots\ldots\ldots + a_n\lambda^n = 0$$

Then the matrix equation

$$a_0 I + a_1 X + a_2 X^2 + \ldots\ldots\ldots\ldots + a_n X^n = 0$$

If the matrix equation is satisfied by A, then

$$a_0 I + a_1 A + a_2 A^2 + \ldots\ldots\ldots\ldots + a_n A^n = 0$$

Since each element of the characteristic matrix $(A - \lambda I)$ is an ordinary polynomial of degree n then the cofactor of every element of $(A - \lambda I)$ i an ordinary polynomial of degree(n-1). Therefore each element of $B = adj\ (A - \lambda I)$ is an ordinary polynomial of degree (n-1).

$\therefore$ We can write

$$B = adj\ (A - \lambda I) = \quad B_0 + B_1\lambda + B_2\lambda^2 + \ldots\ldots\ldots\ldots + B_{n-1}\lambda^{n-1}$$

Where $B_0, B_1$ , $B_{n-1}$ are all square matrices of the same order n whose elements are polynomials in the elements of the square matrix A. We have,

$$(A - \lambda I)\ adj\ (A - \lambda I) = |A - \lambda I| \quad I$$

$$(A - \lambda I)\ B_0 + B_1\lambda + B_2\lambda^2 + \ldots\ldots\ldots\ldots + B_{n-1}\lambda^{n-1} \ = = \quad a_0 + a_1\lambda + a_2\lambda^2 + \ldots\ldots\ldots + a_n\lambda^n)\ I$$

Comparing the coefficient of like powers of $\lambda$ on both sides we get,

$$AB_0 = a_0\ I$$

$$AB_1 - B_0 = a_1\ I$$

$AB_2 - B_1 = a_2 I$

.

.

.

$AB_{n-1} - B_{n-2} = a_{n-1} I$

$-B_{n-1} = a_n I$

Now pre multiplying these equations by $I$, $A$, $A^2$, …….. $A^n$ and then adding we get

$$a_0 I + a_1 A + a_2 A^2 + \ldots\ldots\ldots\ldots + a_n A^n = 0$$

## Example:

Find the characteristic equation of the matrix A = $\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$ and verify that it is satisfied by A. Hence find the inverse of A.

## Solution :

$$|A - \lambda I| = \begin{vmatrix} 2-\lambda & -1 & 1 \\ -1 & 2-\lambda & -1 \\ 1 & -1 & 2-\lambda \end{vmatrix} = 0$$

$$-\lambda^3 + 6\lambda^2 - 9\lambda + 4 = 0$$

$$\lambda^3 - 6\lambda^2 + 9\lambda - 4 = 0$$

This is the required characteristic equation of A. If the characteristic equation is satisfied by A , we must have $A^3 - 6A^2 + 9A - 4I = 0$

$$A^2 = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix}$$

$$A^3 = A^2 . A$$

$$= \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix} \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 22 & -21 & 21 \\ -21 & 22 & -21 \\ 21 & -21 & 22 \end{bmatrix}$$

So that the equation $A^3$ - 6$A^2$ +9A – 4I = 0 become

$$= \begin{bmatrix} 22 & -21 & 21 \\ -21 & 22 & -21 \\ 21 & -21 & 22 \end{bmatrix} - 6 \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix} + 9 \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} - 4 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

This verifies the Cayley – Hamilton theorem.

To find $A^{-1,}$ $A^3$ - 6$A^2$ +9A – 4I = 0

4I = $A^3$ - 6$A^2$ +9A

4I = A($A^2$ - 6$A^1$ +9)

$\frac{I}{A} = \frac{1}{4}$ ($A^2$ - 6$A^1$ +9)

$$A^{-1,} = \frac{1}{4} (A^2 - 6A^1 + 9I)$$

$$= \frac{1}{4} \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix} - \frac{6}{4} \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} + \frac{9}{4} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 3/4 & 1/4 & -1/4 \\ 1/4 & 3/4 & 1/4 \\ -1/4 & 1/4 & 3/4 \end{bmatrix}$$

## 3.8 Diagonalization:

To reduce a given square matrix $A$ to diagonal form, evaluate the characteristic roots ( or eigen values) $\lambda_1$, , $\lambda_2$, …… $\lambda n$ from the characteristic equation of the matrix A. Then the required diagonal matrix D of A can be obtained as the following method.

$$D = \begin{bmatrix} \lambda_1 & 0 & . & 0 \\ 0 & \lambda_2 & . & 0 \\ . & . & . & 0 \\ 0 & 0 & 0 & \lambda_n \end{bmatrix}$$

**Example:**

Diagonalize the matrix $\begin{bmatrix} \frac{4}{3} & \frac{\sqrt{2}}{3} \\ \frac{\sqrt{2}}{3} & \frac{5}{3} \end{bmatrix}$

**Solution:**

Let A = $\begin{bmatrix} \frac{4}{3} & \frac{\sqrt{2}}{3} \\ \frac{\sqrt{2}}{3} & \frac{5}{3} \end{bmatrix}$ and the characteristic equation is | A - $\lambda I$ | = $\begin{vmatrix} \frac{4}{3} - \lambda & \frac{\sqrt{2}}{3} \\ \frac{\sqrt{2}}{3} & \frac{5}{3} - \lambda \end{vmatrix}$

$$\left(\frac{4}{3} - \lambda\right)\left(\frac{5}{3} - \lambda\right) - \frac{\sqrt{2}}{3} . \frac{\sqrt{2}}{3} = 0$$

$$\lambda^2 - 3\lambda + 2 = 0$$

$$(\lambda - 1)(\lambda - 2) = 0$$

Therefore, $\lambda = 1$ and $\lambda = 2$

Then the required diagonal matrix is D = $\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$

**Example:**

Diagonalize the matrix $\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$

**Solution:**

Let A = $\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$

And characteristic equation $|A - \lambda\ I|$ = $\begin{bmatrix} \cos\theta - \lambda & -\sin\theta & 0 \\ \sin\theta & \cos\theta - \lambda & 0 \\ 0 & 0 & 1 - \lambda \end{bmatrix}$ = 0

$$(\cos\theta - \lambda)((\cos\theta - \lambda)(1 - \lambda) - 0) + \sin\theta\,(\sin\theta\,(1 - \lambda) - 0) = 0$$

$$(1 - \lambda)((\cos\theta - \lambda)^2 + sin^2\,\theta) = 0$$

$$(1 - \lambda)(\lambda^2 + 2\lambda\, cos\,\theta + 1) = 0$$

The roots are $\boldsymbol{\lambda}$ **= 1** and $\boldsymbol{\lambda} = \dfrac{\mathbf{2\cos\theta \pm \sqrt{(4\cos^2\theta - 4)}}}{\mathbf{2}}$

I.e., $\lambda = 1$ and $\lambda = \cos\theta \pm \sin\theta$;

$\lambda = 1$ and $\lambda = e^{\pm i\theta}$

Then the eigen values $\lambda_1 = 1$, $\lambda_2 = e^{i\theta}$ , $\lambda_3 = e^{-i\theta}$

The diagonal matrix is $\begin{bmatrix} 1 & 0 & 0 \\ 0 & e^{i\theta} & 0 \\ 0 & 0 & e^{-i\theta} \end{bmatrix}$

**UNIT IV: FOURIER TRANSFORMS & LAPLACE TRANSFORMS**

Definitions -Fourier transform and its inverse - Transform of Gaussian function and Dirac delta function -Fourier transform of derivatives - Cosine and sine transforms - Convolution theorem. Application: Diffusion equation: Flow of heat in an infinite and in a semi - infinite medium - Wave equation: Vibration of an infinite string and of a semi - infinite string. Laplace transform and its inverse - Transforms of derivatives and integrals – Differentiation and integration of transforms - Dirac delta functions -Application - Laplace equation: Potential problem in a semi - infinite strip

## 4.1 Introduction:

Integral transforms are used in the solution of partial differential equations. The choice of a particular transform to be used for the solution of a differential equations depends upon the nature of the boundary conditions of the equation and the facility with which the transform F(s) can be converted to give f(x).

## 4.2 Fourier transform and its inverse:

Fourier Complex transform with the kernel k (s, k) = $e^{-isk}$

$$F[\, f(x) \,] = F(s) = \frac{1}{\sqrt{2\Pi}} \int_{-\infty}^{\infty} f(x) e^{isk}\, dx$$

$$f(x) = \frac{1}{\sqrt{2\Pi}} \int_{-\infty}^{\infty} f(x) e^{-isk}\, dx \quad \text{(Inversion formula)}$$

## 4.3 Fourier Intergral theorem:

It states that $f(x) = \frac{1}{\pi} \int_0^{\infty} dt \int_{-\infty}^{\infty} f\,(t) \cos u(t-x)\, d\, u$

**Proof.**

We know that Fourier series of a function $f(x)$ in $(-c, c)$ is given

$$f(x) = \frac{a_0}{2} + \sum_{n=1} a_n \cos\frac{n\pi x}{c} + \sum_{n=1} b_n \cos\frac{n\pi x}{c} \quad \ldots\ldots\ldots\ldots\ldots(1)$$

where $a0$, $an$ and $bn$ are given by

$$a_0 = \frac{1}{c}\int_{-c}^{c} f(t)dt$$

$$a_n = \frac{1}{c}\int_{-c}^{c} f(t)\ cos\frac{n\pi t}{c}\ dt$$

$$b_n = \frac{1}{c}\int_{-c}^{c} f(t)\ sin\frac{n\pi t}{c}\ dt$$

Substituting the values of $a0$ , $an$ and $bn$ in (1) we get

f(x)=$\frac{1}{2c}\int_{-c}^{c} f(t)dt$+$\sum_{n=1}^{\infty}\frac{1}{c}\ 1\int_{-c}^{c} f(t)\ cos\frac{n\pi t}{c}\ cos\frac{n\pi x}{c}\,dt$+$\sum_{n=1}^{\infty}\frac{1}{c}\ 1$

$\int_{-c}^{c} f(t)\ sin\frac{n\pi t}{c}\ sin\frac{n\pi x}{c}\ \ dt$

$$=\frac{1}{2c}\int_{-c}^{c} f(t)dt + \sum_{n=1}^{\infty}\frac{1}{c}\ 1\int_{-c}^{c} f(t)\ [cos\frac{n\pi t}{c}\ cos\frac{n\pi x}{c} + sin\frac{n\pi t}{c}\ sin\frac{n\pi x}{c}]\ dt$$

$$=\frac{1}{2c}\int_{-c}^{c} f(t)dt + \sum_{n=1}^{\infty}\frac{1}{c}\ 1\ f(t)\ cos\frac{n\pi}{c}\text{(t-x) dt} \qquad \text{-----------(2)}$$

Since cosine functions are even functions i.e., cos(-theta) = cos theta the expression

$$1+2\sum_{n=1}^{\infty} cos\frac{n\pi}{c}(t-x) \ = \ \sum_{n=-\infty}^{\infty} cos\frac{n\pi}{c}(t-x)$$

Therefore, (2) becomes

$$\text{f(x)} = \frac{1}{2c}\int_{-c}^{c} f(t)\ \ \{\sum_{n=-\infty}^{\infty} cos\frac{n\pi}{c}(t-x)\}\ dt$$

$$=\frac{1}{2\pi}\ \int_{-c}^{c} f(t)\ \ \{\ \frac{\Pi}{c}\sum_{n=-\infty}^{\infty} cos\frac{n\pi}{c}(t-x)\}\ \text{dt}$$

Let us now assume that c increases indefinitely, so that we may write $\frac{n\pi}{c} = u$ ,

$\frac{\pi}{c} = u$ . This assumption gives,

$$\lim_{C\to\infty}\ \{\ \frac{\Pi}{c}\sum_{n=-\infty}^{\infty} cos\frac{n\pi}{c}(t-x)\}\ =\ \int_{-\infty}^{\infty}\cos u\ (t-x)\ du \qquad \text{…………..(4)}$$

$$= \int_0^\infty \cos u\,(t-x)\ du$$

Substituting in (3) from (4) we obtain

$$f(x) = \frac{1}{2c}\int_{-\infty}^{\infty} f(t)\ \{\,2\int_0^\infty \cos u\,(t-x)du\}\ dt \qquad \text{--------------(5)}$$

Thus $\quad f(x) = \frac{1}{\pi}\int_0^\infty dt \int_{-\infty}^{\infty} f\ (t)\ \cos u(t-x)\ d\,u$ **Proved.**

**Note:** we have assumed the following conditions on f(x)

(i) f(x) is defined as single valued except at finite points in (-c,c)

(ii) f(x) is periodic outside (-c,c) with period 2c.

(iii) f(x) and f'(x) are sectionally continuous in (-c,c)

## <u>4.4 Fourier sine and cosine integrals</u>:

$$f(x) = \frac{2}{\pi}\int_0^\infty \sin ux\ du \int_{-\infty}^{\infty} f(t)\sin ut\ dt \qquad \textbf{(Fourier sine Integral)}$$

$$f(x) = \frac{2}{\pi}\int_0^\infty \cos ux\ du \int_{-\infty}^{\infty} f(t)\cos ut\ dt \qquad \textbf{(Fourier cosine Integral)}$$

**Proof:**

We know that, cos u (t - x) = cos ( ut - ux)

Cos u (t - x) = cos ut cos ux + sin ut sin ux

Then equation (5)

$$f(x) = \frac{2}{\pi}\int_0^\infty \int_{-\infty}^{\infty} f(t)(\cos ut\cos ux)\ du\ dt + \frac{1}{\pi}\int_0^\infty \int_{-\infty}^{\infty} f(t)(\sin ut\sin ux)\ du\ dt \ \text{........(6)}$$

**Case 1**

When f(t) is off.

f(t) cos ut is odd hence $\int_0^\infty \int_{-\infty}^{\infty} f(t)(\cos ut\cos ux)\ du\ dt = 0$

for odd function $\int_{-a}^{a} f(x)dx = 0$

for even function $\int_{-a}^{a} f(x)dx = 2\int_0^a f(x)dx$

From (6) we have

$$f(x) = \frac{2}{\pi}\int_0^\infty \sin ux\ du \int_0^\infty f(t)\sin ut\ dt \qquad \text{................(7)}$$

The relation (7) is called **Fourier sine integral**.

**Case 2:**

When f(t) is even.

$$\int_0^\infty \int_{-\infty}^\infty f(t)(\sin ut \sin ux) \text{ du dt} = 0$$

f(t) cos ut is even.

From (6) we have

$$f(x) = \frac{2}{\pi} \int_0^\infty \cos ux\, du \int_0^\infty f(t) \cos ut\, dt \qquad \text{……………..(8)}$$

This relation is known as **Fourier cosine integral.**

## 4.5 Fourier sine and cosine Transforms:

$$f(x) = \frac{2}{\pi} \int_0^\infty \sin sx\, ds \int_0^\infty f(t) \sin st\, dt$$

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty \sin sx\, ds\, F(s) \qquad \text{-----------------(1)}$$

$$F(s) = \sqrt{\frac{2}{\pi}} \int_0^\infty \sin sx\, f(t) \qquad \text{------------------(2)}$$

In eqn (2) F(s) is called **Fourier sine transform** of $f(x)$.

In equation (1), $f(x)$ is called the **Inverse Fourier sine transform** of $F(s)$

From equation (8) we have

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty \cos sx\, ds\, F(s) \qquad \text{-----------(3)}$$

$$F(s) = \sqrt{\frac{2}{\pi}} \int_0^\infty \cos sx\, f(t) \qquad \text{-------------(4)}$$

In equation (4), $F(s)$ is called **Fourier cosine transform** of $F(x)$.

In equation (3), $f(x)$ is called the **inverse Fourier cosine transform of** $F(s)$.

### Example

Find the Fourier transform of $\begin{cases} 1 & \text{for } |x| < a \\ 0 & \text{for } |x| > a \end{cases}$

**Solution:**

The Fourier transform of a function $f\,(x)$ is given by

$$F(s) = \frac{1}{\sqrt{2\Pi}} \int_{-\infty}^{\infty} f(x) e^{isk}\, dx$$

Substituting the value of $f\,(x)$, we get

$$F(s) = \frac{1}{\sqrt{2\Pi}} \int_{-a}^{a} 1.\, e^{isk}\, dx = \left[\frac{e^{isk}}{is}\right]_{-a}^{a}$$

$$= \frac{1}{\sqrt{2\Pi}} . \frac{2}{s} . \frac{e^{ias} - e^{-ias}}{2i}$$

$$= \frac{1}{\sqrt{2\Pi}} \frac{2 \sin sa}{s}$$

$$= \sqrt{\frac{2}{\pi}} \frac{\sin sa}{s}$$

## Example:

Find the Fourier sine and cosine transforms of f (x) = $e^{-\alpha x}$ .

**Solution.**

The Fourier sine transform of $f\,(x)$ is given by

$$F(s) = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} \sin sx\, f(x)\, dx$$

Putting the value of $f\,(x)$ we get

$$F(s) = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} e^{-ax} \sin sx\, dx$$

$$= \sqrt{\frac{2}{\pi}} \frac{e^{2} - ax}{a^2 + s^2} \left[ -a \sin sx - s \cos sx \right]_0^{\infty}$$

$$= \sqrt{\frac{2}{\pi}} \left[ -0 + \frac{1}{a^2 + s^2} \text{ x s} \right]$$

The Fourier cosine transform is

$$F(s) = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} e^{-ax} \cos sx\, dx$$

$$= \sqrt{\frac{2}{\pi}} \frac{e^{-ax}}{a^2 + s^2} \left[ -a \cos sx + s \sin sx \right]_0^\infty$$

$$= \sqrt{\frac{2}{\pi}} \frac{a}{a^2+s^2}$$

## 4.6 Dirac delta function:

**Example:**

Find the complex Fourier transform of dirac delta function δ(t – a).

**Solution:**

$$F\{\delta(t-a)\} = \frac{1}{\sqrt{2\Pi}} \int_{-\infty}^{\infty} e^{ist} \, d(t - a) \, dx$$

$$= \frac{1}{\sqrt{2\Pi}} \lim_{h\to 0} \int_{a}^{a+h} \frac{1}{h} e^{ist} \text{ dt}$$

$$= \frac{1}{\sqrt{2\Pi}} \lim_{h\to 0} \frac{1}{h} \left(\frac{e^{ist}}{is}\right)^{a+h} \text{ dt}$$

$$= \frac{1}{\sqrt{2\Pi}} \lim_{h\to 0} \left(\frac{e^{ist}-1}{ish}\right)^{1}$$

$$= e^{ist} \cdot \frac{1}{\sqrt{2\Pi}}$$

**Note.** Dirac delta function $\delta(t-a)$ is defined as

$$\delta(t - a) = \lim_{h\to 0} \text{I (ht - a) where}$$

$$\text{I (ht - a)} = \frac{1}{h} \qquad \text{for } a < t < a+h$$

$$\text{I (ht - a)} = 0 \qquad \text{for } t < a \text{ and } t > a + h$$

## 4.7 CONVOLUTION

The Convolution of two functions $f(x)$ and $g(x)$ is defined as

$$f(x)*g(x) = \int_{-\infty}^{\infty} f(x)g(x-u)du$$

**Convolution Theorem on Fourier Transform**

The Fourier transform of the convolution of $f(x)$ and $g(x)$ is the product of their Fourier transforms, *i.e.,*

$$F\,[\,f(x)*g(x)] = F\,[\,f(x)]\cdot F\,[g(x)]$$

**Proof.**

We know that

$$[\,f(x)*g(x)] = \frac{1}{\sqrt{2\Pi}}\int_{-\infty}^{\infty} f(u)\,.\,g(x-u)\,du \text{ ------------------(1)}$$

Taking Fourier transform of both sides of (1), we have

$$F\,[\,f(x)*g(x)] = \mathrm{F}\,[\int_{-\infty}^{\infty} f(u)\,.\,g(x-u)\,du]$$

$$= \frac{1}{\sqrt{2\Pi}}\int_{-\infty}^{\infty}\ [\frac{1}{\sqrt{2\Pi}}\int_{-\infty}^{\infty} f(u)\,.\,g(x-u)\,du]\ e^{isk}\ \mathrm{dx}$$

$$= \frac{1}{\sqrt{2\Pi}}\int_{-\infty}^{\infty}\{\,f(u)\,du.Fg\ (x-u)\}$$

$$= \frac{1}{\sqrt{2\Pi}}\int_{-\infty}^{\infty} f(u)\,du\,e^{isu}\ \ G(s)$$

$$= \mathrm{G(s)}\ .\ \frac{1}{\sqrt{2\Pi}}\int_{-\infty}^{\infty} f(u)\ \ e^{isu}\ \ du$$

$$= G(s)\cdot F(s)$$

$$= F(s)\cdot G(s)$$ **Proved.**

By inversion

$$F^{-1}\{F(s)\cdot G(s)\} = f*g = F^{-1}\{F(s)\}*F^{-1}\{G(s)\}$$

## 4.8 Fourier transform of derivatives:

We have already seen that,

$$F\{f^{n}(x)\} = (-is)^{n}\,F(s)$$

(*i*) $\therefore$ $\mathrm{F}\,\frac{\partial^2 u}{\partial x^2} = (\ -\mathrm{is}\ )^2\ \mathrm{F}\ \{\mathrm{u}\ (\ \mathrm{x}\ )\} = -\mathrm{s}^2\ \bar{u}$ where $\bar{u}$ is Fourier transform of *u* w.r.t. *x*.

*(ii)* $\mathrm{F_c}\ \{f^{\,'}(x)\} = -\sqrt{\frac{2}{\pi}}\,\mathrm{f}(\ 0\ ) + \mathrm{s}\ \mathrm{F_s}\ (\mathrm{s})$

$$\text{L.H.S} = \sqrt{\frac{2}{\pi}}\int_{0}^{\infty} f'(s).\cos sx\ dx$$

$$= \sqrt{\frac{2}{\pi}}\int_{0}^{\infty}\cos sx\ d\{f(x)\}$$

$= \sqrt{\frac{2}{\pi}}$ f(x) cos sx + $\sqrt{\frac{2}{\pi}}\, s\, \int_0^{\infty} \sin sx\, \{f(x)\}dx$

= s $F_s$ (s) - $\sqrt{\frac{2}{\pi}}$ f(0)

(*iii*) $Fs\{f\,'(x)\}$ = $\sqrt{\frac{2}{\pi}}\, s\, \int_0^{\infty} \sin sx \text{ d } \{f(x)\}$

= $\sqrt{\frac{2}{\pi}}$ f(x) sin sx - $\sqrt{\frac{2}{\pi}}\, s\, \int_0^{\infty} \cos sx\, \{f(x)\}dx$

= - s $F_s$ (s)

(*iv*) $Fc\{f\,''(x)\}$ = $\sqrt{\frac{2}{\pi}}\, \int_0^{\infty} \cos sx \text{ d } \{f'(x)\}$

= $\sqrt{\frac{2}{\pi}}$ f'(x) cos sx + $\sqrt{\frac{2}{\pi}}\, s\, \int_0^{\infty} \sin sx\, \{f'(x)\}dx$

= $\sqrt{\frac{2}{\pi}}\, f\,'(0)$ + s $F_s$ f ′(x)

= - $s^2$ $F_c$ (s) - $\sqrt{\frac{2}{\pi}}\, f\,'\,(0)$

(*v*) $Fs\{f\,''\}(x)$ = $\sqrt{\frac{2}{\pi}}\, \int_0^{\infty} \sin sx \text{ d}\{f'(x)\}$

= $\sqrt{\frac{2}{\pi}}$ f'(x) sin sx - $\sqrt{\frac{2}{\pi}}\, s\, \int_0^{\infty} \cos sx\, \{f'(x)\}dx$

= - s $F_c$ $f\,'(x)$

= - $s^2$ $F_s$ (s) + $\sqrt{\frac{2}{\pi}}$ f (0)

## 4.8 Application of Fourier Transform: (flow of heat)

Let heat flow along a bar of uniform cross-section, in the direction perpendicular to the cross-section. Take one end of the bar as origin and the direction of heat flow is along *x*-axis.

Let the temperature of the bar at any time *t* at a point *x* distance from the origin be *u (x, t)*. Then the equation of one dimensional heat flow is

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$$

**Example:**

A rod of length l with insulated sides is initially at a uniform temperature u. Its ends are suddenly cooled to 0°C and are kept at that temperature. Prove that the temperature function u (x, t) is given by

$$u\,(x, t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right) e^{\frac{-c^2\pi^2 n^2 t}{t^2}}$$

where bn is determined from the equation

$$U_0 = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)$$

**Solution:**

Let the equation for the conduction of heat be

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \qquad \text{-------------(1)}$$

Let us assume that $u = XT$, where $X$ is a function of $x$ alone and $T$ that of $t$ alone

$$\frac{\partial u}{\partial t} = X\frac{\partial T}{\partial t}$$

$$\text{And} \quad \frac{\partial^2 u}{\partial x^2} = T\frac{\partial^2 X}{\partial x^2}$$

Substituting these values in (1), we get $X\frac{\partial T}{\partial t} = c^2\, T\, \frac{\partial^2 X}{\partial x^2}$

$$\frac{1}{c^2\, T}\frac{\partial T}{\partial t} = \frac{1}{X}\frac{\partial^2 X}{\partial x^2} \qquad \text{-------------(2)}$$

Let each side be equal to a constant $(-p^2)$.

$$\frac{1}{c^2\, T}\frac{\partial T}{\partial t} = -p^2$$

$$\frac{\partial T}{\partial t} = \quad p^2 c^2\, T \text{ -------------------(3)}$$

$$\text{and} \qquad \frac{1}{X}\frac{\partial^2 X}{\partial x^2} = -p^2$$

$$\frac{\partial^2 X}{\partial x^2} + p^2 X = 0 \qquad \text{-----------------(4)}$$

Solving (3) and (4) we have

$$T = c_1 e^{-P^2c^2t} \text{ and } X = c_2 \cos px + c_3 \sin px$$

$$U = c_1 e^{-P^2c^2t}(c_2 \cos px + c_3 \sin px) \quad \text{-------------(5)}$$

Putting $x = 0$, $u = 0$ in (5), we get

$$0 = c_1 e^{-P^2c^2t} (c_2)$$

$$c_2 = 0 \text{ since } c_1 \neq 0$$

(5) becomes $u = c_1 e^{-P^2c^2t} c_3 \sin px$ ---------------(6)

Again putting $x = l$, $u = 0$ in (6), we get

$$0 = c_1 e^{-P^2c^2t} (c_2) \sin pl$$

$$\sin pl = 0 = \sin n\pi$$

$$pl = n\pi$$

$$p = \frac{n\pi}{l}$$

Hence (6) becomes $u = c_1 c_3 e^{-P^2c^2t} \sin\left(\frac{n\pi x}{l}\right)$

$$= b_n e^{\frac{-c^2\pi^2n^2}{l^2}} \sin\left(\frac{n\pi x}{l}\right), \qquad b_n = c_1 c_3$$

This equation satisfies the given conditions for all integral values of n. Hence taking n = 1, 2, 3, ......, the most general solution is

$$u(x, t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right) e^{\frac{-c^2\pi^2n^2t}{t^2}}$$

By initial conditions $u = U_0$ when $t = 0$

$$U_0 = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)$$ **Proved.**

**Example:**

Find the solution of $h^2 \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$

for which u (0, t) = u (l, t) = 0, u (x, 0) = $\sin\left(\frac{\pi x}{l}\right)$ by method of variables separable.

**Solution:**

$$h^2 \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \qquad \text{---------(1)}$$

we know that, $\dfrac{\partial^2 u}{\partial x^2} = \dfrac{1}{c^2}\dfrac{\partial u}{\partial t}$ ------------(2)

On comparing (1) and (2) we get

$$h^2 = \frac{1}{c^2}$$

Thus solution of (1) is

$$u = c_1 e^{-\frac{p^2 t}{h^2}} (c_2 \cos px + c_3 \sin px) \qquad \text{-------------(3)}$$

On putting $x = 0$, $u = 0$ in (3) we get $0 = c_1 e^{-\frac{p^2 t}{h^2}} (c_2)$ $\quad c_1 \neq 0$, $c2 = 0$

(3) is reduced to

$$u = c_3 e^{-\frac{p^2 t}{h^2}} c_1 \sin px \qquad \text{-------------------(4)}$$

On putting $x = l$ and $u = 0$ in (4), we get

$$0 = c_3 e^{-\frac{p^2 t}{h^2}} c_1 \sin pl$$

Now (4) is reduced to

$$u = c_1 c_3 e^{-\frac{n^2 \pi^2 t}{h^2 l^2}} \sin\left(\frac{n\pi x}{l}\right) \qquad \text{------------------(5)}$$

On putting $t = 0$, $u = \sin \frac{\pi x}{l}$ in (5) we get

$$\sin \frac{\pi x}{l} = c_4 \sin \frac{n\pi x}{l}$$

This equation will be satisfied if

$$n = 1 \text{ and } c_4 = 1$$

On putting the values of $c_4$ and $n$ in (5), we have

$$u = \sin \frac{\pi x}{l}\, e^{-\frac{\pi^2 t}{h^2 l^2}}$$

## 4.9 Application of Fourier Transform: (Vibration in a infinite string)

Consider an infinitely long freely vibrating string, let y be the displacement of vibration from its mean position and satisfies the wave equation

$$\frac{d^2y}{dx^2} = \frac{1}{v^2}\frac{d^2y}{dt^2} \quad \text{---------------(1)}$$

Where $x$ is the distance measured along the String;

$v$ is the velocity of wave moving along the string: and $y$ is a function $x$ and $t$

The initial condition of the string is $y\,(x, 0\,) = x$

Multiplying on both sides of equation (1) by $\frac{e^{isx}}{\sqrt{2\Pi}}$ and integrating over the limit (-∞, ∞) we get

$$\frac{1}{\sqrt{2\Pi}}\int_{-\infty}^{\infty}\frac{d^2y}{dx^2}\,e^{isk}\,dx = \frac{1}{v^2}\frac{1}{\sqrt{2\Pi}}\int_{-\infty}^{\infty}\frac{d^2y}{dt^2}\,e^{isk}\,dx \quad \text{-------------(2)}$$

It is the Fourier Transform of second derivative
Let

$$Y\,(s,) = \frac{1}{\sqrt{2\Pi}}\int_{-\infty}^{\infty}\frac{y}{1}\,e^{isk}\,dx \quad \text{---------------------(3)}$$

$$\frac{1}{\sqrt{2\Pi}}\int_{-\infty}^{\infty}\frac{d^2y}{dt^2}\,e^{isk}\,dx = (-is)^2\,Y\,(s,t\,) \quad \text{------------(4)}$$

Equation (2) becomes $(-is)^2\,Y\,(s,t\,) = \frac{1}{v^2}\frac{d^2Y(s\,,t)}{dt^2}$ ----------(5)

$$\frac{d^2y}{dt^2} = -\,v^2 s^2\,Y \quad \text{-------------(6)}$$

at t = 0, equation (3) becomes $Y\,(s,0) = \frac{1}{\sqrt{2\Pi}}\int_{-\infty}^{\infty}\frac{y}{1}(x,0)\,e^{\,isk}\,dx$

$$= \frac{1}{\sqrt{2\Pi}}\int_{-\infty}^{\infty}F\,(x)\,e^{\,isk}\,dx = f(s) \quad \text{--------(6)}$$

A general solution of equation (6) is $(s\,,t\,) = f(\,s\,)e^{\,ivst}$ ------------------------( 8)

The inverse Fourier Transform of (3) is

$$y\,(x,t) = \frac{1}{\sqrt{2\Pi}}\int_{-\infty}^{\infty}Y(s,t)\,e^{\,-isk}\,ds \quad \text{------------(9)}$$

$Using\ (8)\ in\ (9),\ We\ get\ y\ (x,t)$

$$= \frac{1}{\sqrt{2\Pi}} \int_{-\infty}^{\infty} f(s)\ e^{-isk}\ ds$$

$$= \frac{1}{\sqrt{2\Pi}} \int_{-\infty}^{\infty} f(s)\ e^{-is(x+vt)}\ ds$$

$$At\ t = 0, we\ have\ (x, 0\ ) = \frac{1}{\sqrt{2\Pi}} \int_{-\infty}^{\infty} f(s)\quad e^{-isx}\ ds\ =\ F(x)$$

$(x)$is the inverse Fourier transform of $f(s)$, therefore $y\ x, t = F(x \pm vt)$. This corresponds to the waves moving in + $x$ and – $x$ directions respectively.

## 4.10 Laplace Transform:

If $F\ (\ t\ )$ be a function of $t$ defined for all values, then Laplace transform of $F\ (t)$ is denoted by $\mathcal{L}(\ F(\ t))$ or $F(\ s\ )$ or $f(\ s\ )$ is defined as

$$\mathcal{L}\ (\mathrm{F(t\ )}) = \mathrm{F(\ s\ )} = f\ (s) =\ \int_{-\infty}^{\infty} F(t)\ e^{-st}\ dt$$

The parameter $s$ is real positive number and the integral exists.

If the integral converges for some value of $(s)$, then only the Laplace transformation of $F(\ t)$ exists otherwise not. $\mathcal{L}$ is Laplace transformation operator. The operation of multiplying $F\ (\ \mathrm{s}\ )$ by $e^{-st}$ and then integrating between the limits 0 to ∞ is known as Laplace transformation.

## First Shifting Theorem:

$$If\ \mathcal{L}(F(t)) = f\ (s\ ), then\ \mathcal{L}\ e^{at}F\ (t) = f(\ s - a)$$

ie., if$f\ s$ is the Laplace transformation of the function $F\ t$ and $a$ is any real or complex number then $f\ (s - a)$ is Laplace transformation of $e^{at}F(\ t)$ .

$$\mathrm{F}(\ s) = \mathcal{L}(\ F(\ t)) \Longrightarrow f\ (s-\ a) = \mathcal{L}\ e^{at}\ F(\ t)$$

**Proof:**

$$\mathcal{L}\ (\mathrm{F(s - a\ )})\ = \int_0^{\infty} \mathrm{F(\ t)}\ e^{-st}\,dt.\ \mathcal{L}\ (e^{at}\ (\mathrm{F(\ t\ )})) =\ \int_0^{\infty} \mathrm{F(\ t)}\ e^{at} e^{-st}\ dt$$

$$=\ \int_0^{\infty} \mathrm{F(\ t)}\ e^{-(-s-a)t}\,dt$$

$$Put\ (s - a)\ u > 0, \qquad =\ \int_0^{\infty} \mathrm{F(\ t)}\ e^{ut}\,dt$$

$= \mathrm{f}\ (\mathrm{u})$

*Replace u by* $f\ (s - a)$ *, then* $L\ e^{at} F\ (t) = f(\ s - a)$ *hence proved*.

## Second Shifting Theorem:

If $\mathcal{L}\ (F(\ t)) = f(\ s\ )$ and $G(\ t\ ) = \begin{cases} \mathrm{F(t-a)} & \mathrm{t} > 0 \\ 0 & \mathrm{t} < a \end{cases}$ then $L\ (G(\ t\ )) = e^{-as} f\ (s)$

**Proof :**

*By definition,* $\mathcal{L}\ (\mathrm{F(t\ )}) = \int_0^\infty \mathrm{G(\ t)}\ e^{-st}\, \mathrm{dt}$

$$= \int_0^a \mathrm{G(\ t)}\ e^{-st}\, \mathrm{dt} + \int_a^\infty \mathrm{G(\ t)}\ e^{-st}\, \mathrm{dt} \quad 0 < \mathrm{a} < \infty$$

$$= \int_0^a 0\ e^{-st}\, \mathrm{dt} + \int_a^\infty \mathrm{F(\ t-a)}\ e^{-st}\, \mathrm{dt}$$

$$= \int_a^\infty \mathrm{F(\ t-a)}\ e^{-st}\, \mathrm{dt}$$

*Put* $\quad t - a = ;\ t = u + a;\ dt = du$

When $\quad u = 0\ ,\ t = a\ and\ u = \infty\ ,\ t = \infty$

$$\therefore \mathcal{L}\ (\mathrm{F(t\ )}) = \int_0^\infty \mathrm{F(\ u)}\ e^{-s(u+a)}\, \mathrm{dt}$$

$$= e^{-sa} \int_0^\infty \mathrm{F(\ u)}\ e^{-su}\, \mathrm{dt}$$

*By properties of definite integrals we can write,* $\mathcal{L}(\ G(\ t\ )) = e^{-sa} \int_0^\infty \mathrm{F(\ t)}\ e^{-st}\, \mathrm{dt}$

$$= e^{-sa} L\ (\mathrm{F(t\ )}) = e^{-sa} f(\ s\ )$$

hence proved

## 4.11 Laplace Transform of derivatives:

If $\mathcal{L}\,(F(\,t\,))= f(\,s\,)\; then\; \mathcal{L}(\,F'(\,t\,))= sf(\,s\,)- F(0)$; if $F(\,t\,)$is continuous for $0 \le t \le N$ and of exponential order for $t > N$ while $F'(\,t\,)$is sectionaly continuous for $0 \le t \le N$.

**Proof:**

**Case1**

$If\ F'\ (t)\ is\ continuous\ for\ all\ t \ge 0\ then = \int_0^\infty \mathrm{F}'(\,\mathrm{t})\ e^{-\mathrm{st}}\,\mathrm{dt}$

$$= [F(t)\ \mathrm{e}^\wedge - \mathrm{st}]_0^\infty - \int_0^\infty \mathrm{F}(\,\mathrm{t})(-\mathrm{s}\ e^{-\mathrm{st}})\mathrm{dt}$$

$$= \lim_{t\to\infty}(\,e^{-\mathrm{st}}\mathrm{F}(\,\mathrm{t})) - \mathrm{F}(0) + \mathrm{s}\int_0^\infty \mathrm{F}(\,\mathrm{t})(\,e^{-\mathrm{st}})\mathrm{dt}$$

$$= \lim_{t\to\infty}(\,e^{-\mathrm{st}}\mathrm{F}(\,\mathrm{t})) - \mathrm{F}(0) + \mathrm{s}\ \mathrm{L}(\mathrm{F}(\mathrm{t}))$$

$$\lim_{t\to\infty}(\,e^{-\mathrm{st}}\mathrm{F}(\,\mathrm{t})) = 0, \quad \text{for s} > \text{a}$$

$$\mathcal{L}(\,F'(\,t\,))= s\ L(\,F(\,t\,))- F(\,0\,)$$

**Case2**

(i) If $F'(\,t\,)$is merely piecewise continuous, then the integral can be broken into sum of integrals in different ranges from $0\ to\ \infty$ such that in each of such parts $F'\ (t)$ is continuous

We have $\mathcal{L}(\,F'(\,t\,))= sL(\,F(\,t\,))- F(\,0\,)$ and $\mathcal{L}(\,G'(\,t\,))= s\mathcal{L}(\,G(\,t)) - G(\,0\,)$

Put $\quad G(t) = F'(t)\ , F''\ (t) = s\mathcal{L}(\,F'(\,t\,))- F'(\,0)$

$$= s(\,s(\,\mathcal{L}(\,F(\,t)) - F(\,0)) - F'(\,0)$$

$$= s^2\mathcal{L}\,(F(\,t\,))- sF(\,0\,)- F'(0) = \mathcal{L}(\,F''(\,t))$$

(ii) $\mathcal{L}\,(H''(\,t)) = s^2\mathcal{L}(\,H(\,t)) - sH(\,0\,)- H'(0)$

Put $\quad (t) = F'(t)\ ,$

$$\mathcal{L}\,(F'''(\,t\,))= s^2\mathcal{L}\,(F'(\,t)) - sF'(\,0\,)- F''(0)$$

$$= s^2\,(s\,(\,\mathcal{L}\,(\,F(\,t)) - F(\,0\,))- sF'(\,0\,)- F''(0)$$

$$\mathcal{L}\,F'''(\,t\,)= s^3\mathcal{L}\,(F(\,t)) - s^2F(\,0\,)- sF'\,(0\,)- F''(0)$$

(iii) If $F'(t)$ and its first (n-1) derivatives are continuous, then proceeding as above we have the general case,

$\mathcal{L}\,(F^n(\,t)) = s^n\,\mathcal{L}(\,F(\,t\,)) - s^{n-1}F(\,0\,) - s^{n-2}\,F'(\,0\,) - \cdots\cdots\cdots\cdots - F^{\,n-1}(0)$

## 4.12 Laplace Transform of Integral:

If $\mathcal{L}\,(F(\,t\,)) = f(\,s\,)$ *then* $\frac{1}{s} f(\,s\,) = \mathcal{L}\,(\,\int_0^t F(u)\,du$

**Proof:**

Let $(\,\mathrm{u}\,) = \int_0^t F(u)\,du$ then

$(\,0\,) = \int_0^t F(u)\,du = 0$

And $G'(\,t\,) = \frac{d}{dt}(\int_0^t F(u)\,du) = \mathrm{F(t)}$

But we know that $\mathcal{L}\,(G'\,(t)) = s\,\mathcal{L}(\,G(\,t\,)) - G(0)$

$\therefore \quad \mathcal{L}\,(t) = s\,\mathcal{L}(\,G(\,t\,)) - (0)$

$\frac{1}{\mathrm{s}}\ \mathcal{L}\,F(t) = \mathcal{L}(\,G(\,t\,)$

$\frac{1}{\mathrm{s}}\ \mathcal{L}\,F(t) = \mathcal{L}\,(\,\int_0^t F(u)\,du\,)$

***Problem*:**

Find $\mathcal{L}\,((\,1\,))$ if Laplace Transform of the function $F(t) = 1$

**Solution:**

We have $\mathcal{L}(\,F(t)) = \int_0^\infty \mathrm{F}(\,\mathrm{t})\,e^{-\mathrm{st}}\,\mathrm{dt}$

$\therefore \quad \mathcal{L}(\,F\,(1)) = \int_0^\infty 1\,e^{-\mathrm{st}}\,\mathrm{dt}$

$= \left[\frac{e-st}{-s}\right]_0^\infty$

$$= \frac{1}{-s}\left(e^{\infty} - e^{0}\right)$$

$$= \frac{1}{-s}(0 - 1)$$

$$= \frac{1}{-s}$$

**Example:**

Find $\mathcal{L}\{ e^{at} \}$

**Solution:**

$$\mathcal{L}( e^{at} ) = \int_0^{\infty} e^{t} \; e^{-st} dt$$

$$= \int_0^{\infty} e^{-(s-a)t} dt$$

$$= \left[\frac{e^{-(s-a)t}}{-(s-a)}\right]_0^{\infty} = e^{\infty} - \frac{1}{-(s-a)}$$

$$= 0 + \frac{1}{(s-a)} = \frac{1}{(s-a)}$$

**Derivatives problems:**

**Example:**

Evaluate (i) $\mathcal{L}(1) = \frac{1}{s}$

(ii) $\mathcal{L}(s) = \frac{1}{s^2}$

(iii) $\mathcal{L}( e^{at}) = \frac{1}{(s-a)}$

**Solution:**

Using Laplace Transform of derivatives

**(i)** $\mathcal{L}(F'(t)) = s\mathcal{L}(F(t)) - F(0)$

Given that (s ) = 1, $\therefore F'(t) = 0, and\ F(0) = 1$

Substituting we get, $(0) = sL(1) - 1$

$$0 = sL(1) - 1$$

$\therefore$L ( 1 )= $\frac{1}{s}$

**(ii)** Given that (s )= $t$, $\therefore F'(t) = 1$, $and\ F(0) = 0$

$\mathcal{L}(1) = s\mathcal{L}(t) - 0,$

but $\mathcal{L}(1) = \frac{1}{s}$

s( t )= $\frac{1}{s}$ and therefore $\mathcal{L}(t) = \frac{1}{s^2}$

**(iii)** $Given\ that$ F(t )= $e^{at}$, $\therefore F'(t) = ae^{at}$, $and\ F(0) = 1$

$Substituting\ we\ get,$ $\mathcal{L}$ (a $e^{at}$ )= $s\mathcal{L}(e^{at}) - 1$

$a\,\mathcal{L}(e^{at}) = s\mathcal{L}(e^{at}) - 1$

$1 = s\mathcal{L}$ (s) $- a\,\mathcal{L}(e^{at})$

$ie.,$ $\mathcal{L}(e^{at})(s - a) = 1$

$\therefore$ $\mathcal{L}(e^{at}) = \frac{1}{(s-a)}$

**Problem of Transform of integrals:**

**Example:**

Evaluate $\mathcal{L}(\int_0^\infty \sin 2u\ \mathrm{du})$

**Solution:**

We have

$$\mathcal{L}(\sin 2) = \frac{2}{(\ s^2+4)} = \mathrm{f(s)}$$

$$\mathcal{L}(F(u)\,du) = \frac{f(s)}{s}$$

$$\therefore \quad \mathcal{L}(\int_0^\infty \sin 2u\ \mathrm{du}) = \frac{2}{s(\ s^2+4)}$$

## 4.13 Inverse Laplace Transform

**Partial fraction method:**

Any rational function $\frac{P(s)}{Q(s)}$ where $(s)$ and $(s)$ are polynomials with the degree of $(s)$ less than that of $Q(s)$ can be written as the sum of rational functions (called partial fraction) having the form $\frac{A}{(as+b)^\wedge r}$, $\frac{As+B}{(as^2+bs+c)^r}$ where r = 1,2,3,……… By finding the inverse Laplace transform of each of the partial fractions we can find $L^{-1}\frac{P(s)}{Q(s)}$

**Example:**

1. $\frac{2s-5}{(3s-4)(2s+1)^2}$

$$= \frac{A}{3s-4} + \frac{B}{(2s+1)^3} + \frac{C}{(2s+1)^2} + \frac{D}{2s+1}$$

2. $\frac{3s^2-4s+2}{(s^2+2s+4)^2\,(s-5)}$

$$= \frac{As+B}{(s^2+2s+4))^2} + \frac{Cs+D}{(s^2+2s+4)^1} + \frac{E}{s-5}$$

**Inverse Laplace Transform definition:**

If the Laplace transform of a function F(t) is f (s)

ie if $L(F(t)) = f(s)$, then $F(t)$ is called an inverse Laplace transform of $f(s)$ .

ie, $F(t) = L^{-1}(f(s))$

Where $L^{-1}$ is called the inverse Laplace transformation operator.

**Example:**

$$Find\ L^{-1}\ \frac{3s+7}{(s^2-2s-4)^1}$$

**Solution:**

$$\frac{3s+7}{(s^2-2s-4)^1} = \frac{3s+7}{(s-3)(s+1)} = \frac{A}{s-3} + \frac{B}{s+1}$$

$$3s + 7 = (+1) + B\,(s-3)$$

$$= A + B\,s + A - 3B$$

Equating the coefficient of $s$ and constant terms we get

$$A + B = 3;\ and\ A - 3B = 7$$

Solving these equations we get, $A = 4\ and\ B = -1$

$$\frac{3s+7}{(s-3)(s+1)} = \frac{4}{s-3} - \frac{1}{s+1}\, L^{-1}\ \frac{3s+7}{(s^2-2s-4)^1}$$

$$= 4\,L^{-1}\ \frac{1}{s-3} - L^{-1}\ \frac{1}{s+1}$$

$$= 4e^{3t} - e^{-t}$$

$Because \qquad L^{-1}\ \frac{1}{s-a} = e^{at}$

$$\mathrm{L}\ e^{\mathrm{at}} = \frac{1}{s-a}$$

**4.14 Dirac delta function**:

We know that

$$\int_0^{\infty} f(t)\ \delta(t-a) = \mathrm{f(a)}$$

Replacing f(t) by $e^{-st}$ in above equation, we get

$$\int_0^{\infty} e^{-st}\ \delta(t-a) = [\ \mathrm{e}^{-\mathrm{st}}\ ]_{t=a}$$

$$= \mathrm{e}^{-\mathrm{sa}}$$

$$\mathrm{L}(\delta(t-a)) = \mathrm{e}^{-\mathrm{sa}}$$

If a = 0, then L $(\delta(t)) = 1$

**Example:**

Find the Laplace transform of $t^3\,\delta(t-4)$

**Solution:**

$$\mathrm{L}(\mathrm{t}^3\,\delta(t-4)) = \int_0^{\infty}\ \mathrm{e}^{-\mathrm{st}}\ t^3\ \delta(t-a)\ \mathrm{dt}$$

$$= 4^3\ \ \mathrm{e}^{-4\mathrm{s}}$$

**Example:**

Find the laplace transform of $e^{-4t}\ \delta(t-3)$

**Solution:**

$$\mathrm{L}(\mathrm{e}^{-4\mathrm{t}}\ \delta(t-3)) = \int_0^{\infty}\ e^{-st}\ \mathrm{e}^{-4\mathrm{t}}\,\delta(t-a)\ \mathrm{dt}$$

$$= \mathrm{e}^{-3(\mathrm{x}+4)}$$

## 4.15 Application:

## Potential problem in semi – infinite strip:

Laplace's equation $\nabla^2 u = 0$ inside a semi-infinite strip ($0 < x < \infty$, $0 < y < H$) with the following boundary conditions:

$$U(x,0) = 0,\ u(x,H) = 0,\ u(0,y) = f(y).$$

using separation of variables

a linear combination of solutions of Laplace's equation of the form X ( x ) Y ( y ) . Then

$$\nabla^2 (XY) = X''(x)\, Y(y) + X(x)\, Y''(y) = 0$$

so that $\frac{X''}{X} + \frac{Y''}{Y} = 0$

The first term on the left is a function only of x and the second a function only of y. The only way this equation can hold for all x and y is if each term is constant. Hence

$$X'' = CXY'' = -CY$$

For some real constant C. the values of C depend on the boundary conditions, which are $X(0) = 1$, $\lim_{x\to\infty} X(x) = Y(0) = Y(h) = 0$

With Y(y) not identically zero.

The easiest boundary condition to satisfy is that $X(x) \to 0$ as $x \to \infty$.

$$X(x) = e^{-kx} \quad \text{for some } k > 0$$

$$Y'' = -k^2 Y$$

Subject to $Y(0) = Y(h) = 0$ but with Y(y) not identically zero. That can be done if $k = \frac{n\pi}{h}$ . n is some positive integer.

$$Y(y) = B \sin\left(\frac{n\pi y}{h}\right)$$

Eigenfunction, $X_n(x)\, Y_n(y) = e^{\frac{-n\pi y}{h}}\, e^{\frac{n\pi y}{h}}$

The linear combination of above equation is

$$u(x,y) = \sum_{n=1}^{\infty} an\, e^{\frac{-n\pi y}{h}}\, e^{\frac{n\pi y}{h}}$$

the coefficient $a_n$ to satisfy the boundary condition $u(0,y) = f(y)$. then

$$f(y) = u\ (o,y) = \sum_{n=1}^{\infty} a_n\ e^{\frac{n\pi y}{h}}$$

which is the fourier sine series for f(y) on the interval $0 \le y \ \le h.$ thus

$$a_n = \frac{2}{h}\int_0^h f(y) \sin\ (\frac{n\pi y}{h})\ dy$$

**UNIT V: DIFFERENTIAL EQUATIONS**

Second order differential equation- Sturm - Liouville's theory - Series solution with simple examples - Hermite polynomials - Generating function - Orthogonality properties - Recurrence relations – Legendre polynomials – Generating function -Rodrigue formula –Orthogonality properties -Dirac delta function - One dimensional Green's function and Reciprocity theorem- Sturm- Liouville's type equation in one dimension & their Green's function.

## 5.1 Second order differential equation :

The general form of the linear differential equation of second order is

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Q\,y = R$$

Where P and Q are constants and R is a function of x or constant

### 5.2 STRUM-LIOUVILLE EQUATION :

$$\frac{d}{dx}\left[p(x)\,.\,\frac{dy}{dx}\right] + [\lambda\, q(x) + r(x)]\,y = 0$$

**Solution.**

We know that Bessel's equation is

$$X^2\frac{d^2y}{dX^2} + X\frac{dy}{dX} + (X^2 - n^2)\,y = 0 \text{ ---------------------(1)}$$

Substituting $X = kx$ in (1), we get

$$\frac{dy}{dX} = \frac{dy}{dx}\frac{dx}{dX} = \frac{dy}{dX}\frac{1}{k} \text{ and } \frac{d^2y}{dX^2} = \frac{d^2y}{dx^2}\frac{1}{k^2}$$

$$k^2\,x^2\left(\frac{d^2y}{dX^2}\frac{1}{k^2}\right) + k\,x\left(\frac{dy}{dx}\frac{1}{k}\right) + (k^2x^2 - n^2)\,y = 0$$

$$x^2\frac{d^2y}{dX^2} + x\frac{dy}{dX} + (k^2x^2 - n^2)\,y = 0$$

$$x\frac{d^2y}{dX^2} + \frac{dy}{dX} + \left(k^2x^2 - \frac{n^2}{x}\right)y = 0$$

$$\left[x\frac{d^2y}{dX^2} + \frac{dy}{dX}\right] + \left(\lambda x - \frac{n^2}{x}\right)y = 0 \qquad (\text{Put } k^2 = \lambda)$$

$$\frac{d}{dx}\left[p(x)\,.\,\frac{dy}{dx}\right] + \left[\lambda\,x - \frac{n^2}{x}\right]y = 0 \qquad \text{-------------(2)}$$

Equations (1) and (2) are of the form.

$$\frac{d}{dx}\ [\, p(x)\ .\ \frac{dy}{dx}\,] + [\lambda\ q\ (x) + r\ (x)]\ y = 0 \ \ldots\ldots\ldots\ldots\ldots\ldots\ (3)$$

Equation (3) is known as the Strum-Liouville equation.

Equation (3) with the following conditions is known as Strum-Liouville problem.

$$\alpha_1\ y\ (a) + \alpha_2\ y'\ (a) = 0$$

$$\beta_1\ y\ (b) + \beta_2\ y'\ (b) = 0$$

Solution of Strum-Liouville problem is called an eigen function where $\lambda$ is an eigen value.

Particular Case. Putting $p = 1$, $q = 1$, $r = 0$ in (3), we have

$$\frac{d^2y}{dX^2} + \lambda\ y = 0$$

Now taking conditionsα a1s = $\beta_1 = 1$ and $\alpha_2 = \beta_2 = 0$

$$y\ (a) = 0 \text{ and } y\ (b) = 0$$

Hence $\qquad y'' + \lambda\ y = 0$

$\qquad\qquad y\ (a) = 0,\ y\ (b) = 0$

These are the simplest form of Strum - Liouville problem

## 5.3 Hermite polynomial:

Hermite differential equation is

$$\frac{d^2y}{d\varkappa^2} - 2x\frac{dy}{dx} + 2ny = 0 \text{ --------------(1)}$$

Where n is positive integer. Solution for this equation is known as Hermite function. To find singular points and series solution of the equation, consider $P_{\ x} = -2x \ and \ Q_{\ x} = 2n$

There are no singular points. By Fuchs theorem, Hermite differential equation has a series Solution

$$y = \sum_{m=0}^{\infty} a_m\ \ x^{k-m}\ a_0 \neq 0 \qquad \text{-------------(2)}$$

On differentiating equation (1) with respect to $x$, we get

$$1\frac{dy}{dx} = \sum_{m=0}^{\infty} a_m\ (k-m)\ \ x^{k-m-1} \qquad \text{--------------(3)}$$

$$\frac{d^2y}{d\varkappa^2} = \sum_{m=0}^{\infty} a_m\ (k-m)\ (k-m-\ 1)\ x^{k-m-2} \qquad \text{------------(4)}$$

Using 1, 2, 3, 4 we get,

$$\sum_{m=0}^{\infty} a_m\,(k-m)\,(k-m-\ 1)\,x^{k-m-2} \ + \sum_{m=0}^{\infty} a_m\, 2\,(n-k+\ m)\,x^{k-m} \ = \ 0 \text{ -----(5)}$$

(5) is polynomial equation. Equating the coefficient of the highest power of $x$ to zero,

ie., $a_0\,(n - k) = 0$

since $a_0 \neq 0$, $k = n$, then equating the coefficient of $xk{-}1$ to zero

,ie., $a_1\,(n-k+1) = 0$

For $k = n$ we have $(n-k+1) \neq 0$ and therefore,

$$a_1 = 0.$$

Further equating the coefficient of $xk\ {-}r$ to zero, we get

$$a_{r-2}\,(k-r+2\,)(k-r+1) + 2a_r\,(n-k+r) = 0$$

$$a_r = -\frac{(k-r+2)(k-r+1)}{2(n-k+r)}\ a_{r-2} \text{ -------------------(6)}$$

Since $a_1 = 0$ equation (6) gives $a_3 = a_5 = a_7 \cdots \cdots = 0$ then for $k = n$, we have

$$a_r = -\frac{(n-r+2)(n-r+1)}{2r}\ a_{r-2}$$

$a_2 = -\ \frac{n(n-1)}{2.2}\,\mathrm{a}_0$

$\mathrm{a}_4 = -\ \frac{(n-2)(n-3)}{2.6}\,\mathrm{a}_2\ = \frac{n(n-1)(n-1)(n-3)}{2^4.3\,!}\,\mathrm{a}_0$

$\mathrm{a}_6 = -\ \frac{(n-4)(n-5)}{2.6}\,\mathrm{a}_4\ = \frac{n(n-1)\ldots\ldots\ldots(n-5)}{2^6.3\,!}\,\mathrm{a}_0$

.

.

.

$\mathrm{a}_{2r} = (-1)^r\ \frac{n\,(n-1)\ldots\ldots..(n-2r+1)}{2^{2r}.6}\,\mathrm{a}_0\ = \frac{(-1)^r n\,!}{2^{2r}.r\,!(n-2r)!}\,\mathrm{a}_0$ -----------------(7)

Let $a_0 = 2^n$ and substitute in (7) we get

$$\mathrm{a}_{2r} = \frac{(-1)^r n\,!}{2^{2r-n}.r\,!(n-2r)!} \text{ ----------------(8)}$$

Since $a_1 = a_3 = a_5 = a_7 \cdots \cdots = 0$ then equation can be written as

$$y = \sum_{r=0}^{\infty} a_{2r}\ x^{n-2r}\ a_0 \neq 0 \text{ -----------------(9)}$$

Substitute (9) in (8) we get solution of Hermite differential equation denoted by $(x)$

$$y = (x) = \sum_{r=0}^{\infty} \frac{(-1)^r n\,!}{2^{2r-n}.r\,!(n-2r)!} X^{n-2r}$$

$$= \sum_{r=0}^{N} \frac{(-1)^r n\,!}{r\,!(n-2r)!} (2X)^{n-2r}$$

**Some Specific cases for $H_n(x)$:**

(i) $H_n(x) = 1$

(ii) $H_1(x) = 2x$

(iii) $H_2(x) = 4x^2 - 2$

(iv) $H_3(x) = 8x^3 - 12x$

(v) $H_4(x) = 16x^4 - 48x^2 + 12$

(vi) $H_5(x) = 32x^5 - 160x^3 + 120x$

(vii) $H_6(x) = 64x^6 - 480x^4 + 720x^2 - 120$

(viii) $H_n(0) = \frac{(-1)^{n/2} n\,!}{(n/2)!}$ when $n$ is even integer

(ix) $H_n(0) = 0$ when $n$ is odd integer

(x) $H_n -x = (-1)^n H_n x$

## 5.4 Generating function for$(x)$:

The function $e^{2xt-t^2}$ is known as generating function for Hermite function. The coefficient of

$t^n$ in the expansion is $\frac{H_n(x)}{n!}$ ie. $e^{2xt-t^2} = \Sigma \frac{H_n(x)}{n!} t^n$

**Proof:**

we know that $e^{2xt} = \sum_{r=0}^{\infty} \frac{(2x)^r\ t^r}{r!}$

And $$e^{-t^{-2}} = \sum_{r=0}^{\infty} \frac{(-1)^s \ t^{2s}}{s!}$$

$$e^{2xt} \ e^{-t\text{^-}2} = \sum_{r=0}^{\infty} \frac{(2x)^r \ t^r}{r!} \sum_{r=0}^{\infty} \frac{(-1)^s \ t^{2s}}{s!}$$

$$e^{2xt-t^2} = \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \frac{(-1)^s (2x)^r \ t^n}{s! \, r!}$$

Put $r = n - 2s$ in above equation we get

$$e^{2xt-t^2} = \sum_{n=2s}^{\infty} \sum_{s=0}^{\infty} \frac{(-1)^s (2x)^{n-2s} \ t^n}{s!(n-2s)!}$$

This shows that the coefficient of $t_n$ of the expansion of $e^{2xt-t^2}$ is $\frac{H_n(x)}{n!}$

## 5.5 Recurrence relations for($x$):

The relations among various orders of the Hermite function are known as recurrence relations

(i) We know that $e^{2xt-t^2} = \Sigma \frac{H_n(x)}{n!} t^n$ -----------------(1)

Differentiating above equation with respect to $t$, we get

$$e^{2xt-t^2}(2x - 2t) = \Sigma \frac{H_n(x)}{n!} n \ t^{n-1} \text{ -----------------(2)}$$

Substitute (2) in (1) we get

$$2x \frac{H_n(x)}{m!} - 2 \frac{H_{m-1}(x)}{m-1!} = \frac{H_{m+1}(x)}{m!}$$

$$2xH_m(x) = H_{m+1}(x) + 2mH_{m-1}(x) \text{ --------------(3)}$$

(ii) Differentiating equation (1) with respect to $x$, we get

$$e^{2xt-t^2} 2t = \Sigma \frac{H'_n(x)}{n!} t^n \text{ -----------------(4)}$$

Substitute equation (1) in (4) we get

$$2t \ \Sigma \frac{H_n(x)}{n!} t = \Sigma \frac{H'_n(x)}{n!} t^n \text{ -----------------(5)}$$

Equating the coefficient of $t^m$ on both of (5) we get

$$2\,\frac{H_{m-1}(x)}{(m-1)!} = \frac{H'_n(x)}{m!} \quad \text{Or}$$

$$2m\, H_{m-1}(x) = H'_m(x) \quad \text{----------------(6)}$$

**(iii)** Equating the coefficient $t_0$ we get $H_m{}'(x) = 0$

**(iv)** Substitute the value of $2mH_{m-1}(x)$ from above equation we get

$$2mH_{m-1}(x) = H_{m+1}(x) + H_m{}'(x)$$

## 5.6 Orthogonality properties:

The orthogonal property of Hermite polynomials is

$$\int_{-\infty}^{\infty} e^{-x^2}\,(x)\,H_n(x)\,dx = 0, \qquad m \neq n$$

$$2^n\, n!\,\sqrt{\pi}\,, \qquad m = n$$

**Solution:**

we know that $\quad e^{x^2-(t_1-x))^2} = \sum \frac{H_n(x)}{n!}$ t1^n (generating function) -----(1)

$$e^{x^2-\ (t_2-x))^2} = \sum \frac{H_n(x)}{n!} \ t\,2\text{^}n \quad \text{------(2)}$$

Multiplying (1) and (2) we get

$$e^{x^2-(t_1-x)^2} . e^{x^2-(t_2-x)^2} = \left[\sum \frac{H_n(x)}{n!}\ t1^n\right]\left[\sum \frac{H_n(x)}{n!}\ t2^n\right]$$

$$= \sum_{\substack{n=0\\m=0}}^{\infty} \left[ \frac{H_n(x)}{n!}\ t1^n \,\Big|\, \sum \frac{H_n(x)}{n!}\ t2^n \right] \frac{t_1^n t_2^m}{n!m!}$$

Multiplying both side of this equation by $e^{-x^2}$ and then integrating with the limits for -∞ to ∞, we have

$$\sum_{\substack{n=0\\m=0}}^{\infty} \int_{-\infty}^{\infty} e^{-x^2} \left[\frac{H_n(x)}{1}\ t1^n \,\Big|\, \sum \frac{H_n(x)}{1}\ t\,2^n \right] \frac{t_1^n t_2^m}{n!m!}$$

$$= e^{-x^2} \int_{-\infty}^{\infty} e^{x^2-(t_1-x)^2} . e^{x^2-(t_2-x)^2}\ .dx$$

$$= \int_{-\infty}^{\infty} e^{(x^2-(t_1-x)^2)-(t_2-x)^2}\ dx$$

$$= e^{-(t_1^2 + t_2^2)} \int_{-\infty}^{\infty} e^{(x^2 - 2x(t_1 + t_2)^1)} \text{ dx} \quad \text{-----------(3)}$$

We know that

$$\int_{-\infty}^{\infty} e^{(-ax^2+2bx)} \text{ dx } = \sqrt{\frac{\pi}{2}} \mathrm{e}^{\frac{b^2}{a}} \quad \text{(Standard formula)} \quad \text{------------(4)}$$

Replacing by (t1 + t2) and by 1 in (4) we get,

$$\int_{-\infty}^{\infty} e^{(x^2-2x(t_1 + t_2)^1)} \text{ dx} = \sqrt{\pi}\; e^{(t_1 + t_2)^2} \text{ ----------------(5)}$$

Putting the value of $\int_{-\infty}^{\infty} e^{(x^2-2x(t_1 + t_2)^1)}$ dx $from$ (5) in R.H.S of (3) we get

$$e^{(t_1 + t_2)^2} . \sqrt{\pi}\; e^{(t_1 + t_2)^2} = \sqrt{\pi}\; e^{-t_1^2 - -t_2^2 + t_1^2 + t_2^2 + 2t_1 t_2} = \sqrt{\pi} e^{2 t_1 t_2}$$

$$= \sqrt{\pi}\,[\ 1 + 2\, t_1 t_2 + \frac{(2\, t_1\, t_2)^2}{2!} + \frac{(2t_1\, t_2)^3}{3!} + \ldots\ldots..]$$

$$= \sqrt{\pi} \sum \frac{(2\, t_1\, t_2)^n}{n!}$$

$$= \sqrt{\pi} \sum \frac{2^n t_1^n t_2^n}{n!}$$

$$= \sqrt{\pi} \sum_{\substack{m=0 \\ n=0}}^{\infty} 2^n t_1^n t_2^n\ \delta_{m.n}$$

From (3) we have

$$\sum_{\substack{n=0 \\ m=0}}^{\infty} \int_{-\infty}^{\infty} e^{-x\text{^}2} \left[\frac{H_n(x)}{1} \quad \mathrm{t1}^n \ \middle|\ \sum \frac{H_n(x)}{1} \quad \mathrm{t2}^n\right] \frac{t_1^n t_2^m}{n!m!} = \sqrt{\pi} \sum \frac{2^n t_1^n t_2^n}{n!}$$

On equating the coefficients of $t_1^n t_2^m$ on both sides, we get

$$\int_{-\infty}^{\infty} e^{-x^2} \frac{Hn(x)Hm(x)}{n!m!} = \sqrt{\pi} \frac{2^n}{n!} \delta_{m.n}$$

$$\int_{-\infty}^{\infty} e^{-x^2} \frac{Hn(x)Hm(x)}{n!m!} = \sqrt{\pi} \frac{2^n}{1} m!\, \delta_{m.n}$$

$$\int_{-\infty}^{\infty} e^{-x^2} \frac{Hn(x)Hm(x)}{n!m!} = \quad 0 \quad , \qquad m \neq n$$

$$\sqrt{\pi}\, 2^n\, m!, \qquad\qquad m = n$$

**Example:**

Find the value of $\int_{-\infty}^{\infty} e^{-x^2} \; H_2(x) \; H_3(x)$ dx

**Solution:**

We know that

$$\int_{-\infty}^{\infty} e^{-x^2} \quad H_2(x) \; H_3(x) \, \mathrm{dx} = 0 \quad \text{if m} \neq n$$

Here m = 2 and n = 3 , m $\neq n$

Hence, $$\int_{-\infty}^{\infty} e^{-x^2} \quad H_2(x) \; H_3(x) \, \mathrm{dx} = 0$$

## 5.7 LEGENDRE'S EQUATION:

Since *Pn* (*x*) and *Qn* (*x*) are two independent solutions of Legendre's equation, therefore the most general solution of Legendre's equation is

$$y = AP_n(x) + B\,Q_n(x)$$

where *A* and *B* are two arbitrary constants.

## 5.8 RODRIGUE'S FORMULA:

$$Pn(x) = \frac{1}{2^n\, n!} \frac{d^n}{dx^n} (x^2 - 1)^n$$

**Proof:**

$$\text{Let } v = (x^2 - 1)^n \quad \text{—————————(1)}$$

Then $$\frac{dy}{dx} = n\,(x^2 - 1)^n \,(2x)$$

Multiplying both sides by $(x^2 - 1)$, we get

$$(x^2 - 1)\frac{dy}{dx} = 2n\,(x^2 - 1)^n \; x$$

$$(x^2 - 1)\frac{dy}{dx} = 2\,n\,v\,x \qquad \text{----------------(2)}$$

Now differentiating (2), $(n + 1)$ times by Leibnitz's theorem, we have

$$(x^2 - 1)\frac{d^{n+2}v}{dx^{n+2}} + (n+1)_{C_1}\,(2x)\frac{d^{n+1}v}{d^n+1} - (n+1)_{C_2}\,(2)\frac{d^n v}{d^\wedge n} = 2n\left[x\frac{d^{n+1}v}{dx^{n+1}} + (n+1)_{C_1}\,(1)\frac{d^{n+1}v}{d^n+1}\right]$$

$$(x^2 - 1)\frac{d^{n+2}v}{dx^{n+2}} + 2x\left[(n+1)_{C_1} - n\right]\frac{d^{n+1}v}{dx^{n+1}} + 2\left[(n+1)_{C_2} - n.(n+1)_{C_1}\right]\frac{d^n v}{dx^\wedge n} = 0$$

$$(x^2 - 1)\frac{d^{n+2}v}{dx^{n+2}} + 2x\frac{d^{n+1}v}{dx^{n+1}} - n(n+1)\frac{d^n v}{dx^\wedge n} = 0 \text{ ------------(3)}$$

If we put $\frac{d^n v}{dx^n} = y$ (3) become,

$$(x^2 - 1)\frac{d^2 y}{dx^2} + 2x\frac{dy}{dx} - n(n+1)y = 0$$

$$(1 - x^2)\frac{d^2 y}{dx^2} - 2x\frac{dy}{dx} + n(n+1)y = 0$$

This shows that $y = \frac{d^n y}{dx^n}$ is a solution of Legendre's equation

$$\therefore\ C\frac{d^n y}{dx^n} = Pn(x) \text{ -----------(4)}$$

where $C$ is a constant.

But $v = (x^2 - 1)^n = (x^1 + 1)^n\ (x^1 - 1)^n$

$$\frac{d^n v}{dx^n} = (x+1)^n\frac{d^n\ (x-1)^n}{dx^n} + n_{C_1}\ .\ n\,(x+1)^n\frac{d^{n-1}}{dx^{n-1}}\,(x+1)^n + \ldots\ldots + (x-1)^n\frac{d^{n-1}}{dx^{n-1}}\,(x+1)^n = 0$$

when $x = 1$, $\qquad \frac{d^n v}{dx^n} = 2^n\,n!$

All the other terms disappear as $(x - 1)$ is a factor in every term except first.

Therefore when $x = 1$, (4) gives

$$C.\ 2^n\,n! = Pn(1) = 1$$

$$C = \frac{1}{2^n\,n!} \qquad \text{-------------(5)}$$

Substituting the value of $C$ from (1) in (5) we have

$$Pn\,(x) = \frac{1}{2^n\,n!}\frac{d^n v}{dx^n}$$

$$Pn\,(x) = \frac{1}{2^n\,n!}\frac{d^n}{dx^n}(x^2 - 1)^n$$

## Example:

Let Pn (x) be the Legendre polynomial of degree n. Show that for any function, f (x), for which the nth derivative is continuous,

$$\int_{-1}^{1} f(x)Pn(x)dx = \frac{(-1)^n}{2^n\, n!}\int_{-1}^{1}(x^2-1)^n \;\; f^n\,(x)dx$$

## Solution:

$$\int_{-1}^{1} f(x)Pn(x)dx = \frac{(-1)^n}{2^n\, n!}\int_{-1}^{1}(x^2-1)^n \;\; f^n\,(x)dx$$

$$Pn\,(x) = \frac{1}{2^n\, n!}\frac{d^n}{dx^n}(x^2-1)^n$$

$$= \frac{1}{2^n\, n!}\int_{-1}^{1} f(x)\,\frac{d^n}{dx^n}(x^2-1)^n\,\mathrm{dx}$$

Integrating by parts, we get

$$= \frac{1}{2^n\, n!}\left[\,\mathrm{f(x)}\,\frac{d^n}{dx^n}(x^2-1)^n - \int f'(x)\,\frac{d^n}{dx^n}(x^2-1)^n\right]_{-1}^{+1}$$

$$= \frac{1}{2^n\, n!}\left[\,0 - \int_{-1}^{1} f'(x)\,\frac{d^{n-1}}{dx^{n-1}}(x^2-1)^n\,\mathrm{dx}\right]$$

$$= \frac{(-1)}{2^n\, n!}\int_{-1}^{1} f'(x)\,\frac{d^{n-1}}{dx^{n-1}}(x^2-1)^n\,\mathrm{dx}$$

Again integrating by parts, we have

$$= \frac{(-1)}{2^n\, n!}\left[\,\mathrm{f'}\,(x)\frac{d^{n-1}}{dx^{n-1}}(x^2-1)^n - \int f''(x)\;\frac{d^{n-2}}{dx^{n-2}}(x^2-1)^n\,dx\right]_{-1}^{+1}$$

$$= \frac{(-1)^\wedge 2}{2^n\, n!}\int_{-1}^{1} f''(x)\,\frac{d^{n-1}}{dx^{n-1}}(x^2-1)^n\,\mathrm{dx}$$

Integrating $(n-2)$ times, by parts, we get

$$= \frac{(-1)^n}{2^n\, n!}\int_{-1}^{1}(x^2-1)^n \;\; f^n\,(x)dx$$ **Proved.**

## 5.9 LEGENDRE POLYNOMIALS:

$$P_n(x) = \frac{1}{2^n\, n!}\frac{d^n}{dx^n}(x^2-1)^n$$

If $n = 0$, $\quad P_0(x) = \frac{1}{2^0\, 0!} = 1$

If $n = 1$, $\quad P_1(x) = \frac{1}{2^1\, 1!}\frac{d^1}{dx^1}(x^2-1)^1 = (1/2)(2x) = x$

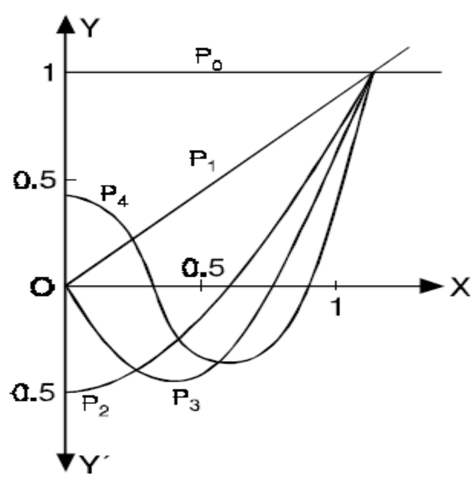
If $n = 2$, $\quad P_2(x) = \frac{1}{2^2\, 2!}\frac{d^2}{dx^2}(x^2-1)^2 = \frac{1}{8}\frac{d}{dx}\left(2(x^2-2)(2x)\right)$

$$= \frac{1}{2}((x^2-1).1 + 2x\,.\,x) = \frac{1}{2}(3x^2-1).$$

similarly $\quad P_3(x) = \frac{1}{2}(5x^3-3x)$.

$$P_4(x) = \frac{1}{8}(35x^4-3x^2+3).$$

$$P_5(x) = \frac{1}{8}(63x^5-70x^3+15x).$$




$$P_n(x) = \sum_{r=0}^{N} \frac{2!}{2^n\,.r\,(n-r)!(n-2r)!}\, x^{n-2r}$$

where $\quad N = \frac{n}{2}$ if $n$ is even.

$$N = \frac{n}{2}(n-1) \text{ if } n \text{ is odd.}$$

**Example:**

Express f (x) = 4 $x^3$ + 6 $x^2$ + 7 x + 2 in terms of Legendre Polynomials.

**Solution.**

Let $4x^3 + 6x^2 + 7x + 2 \equiv a P_3(x) + b P_2(x) + c P_1(x) + d P_0(x)$ ………..(1)

$$= a\left(\frac{5x^3}{2} - \frac{3x}{2}\right) + b\left(\frac{3x^2}{2} - \frac{1}{2}\right) + c\,(x) + d\,(1)$$

$$= \frac{5ax^3}{2} - \frac{3ax}{2} + \frac{5bx^2}{2} - \frac{b}{2} + cx + d$$

$$= \frac{5ax^3}{2} + \frac{5bx^2}{2} + \left(\frac{-3a}{2} + c\right)x - \frac{b}{2} + d$$

Equating the coefficients of like powers of *x*, we have

$$4 = \frac{5a}{2}, \text{ or } a = \frac{8}{5}$$

$$6 = \frac{3b}{2}, \text{ or } b = 4$$

$$7 = \frac{-3a}{2} + c \text{ or } 7 = \frac{-3}{2}\left(\frac{8}{5}\right) + c \text{ or } c = \frac{47}{5}$$

$$2 = \frac{-b}{2} + d \text{ or } 2 = \frac{-4}{2} + d \text{ or } d = 4$$

Putting the values of *a*, *b*, *c*, *d* in (1), we get

$$4x^3 + 6x^2 + 7x + 2 = \frac{8}{5} P_3(x) + 4 P_2(x) + \frac{47}{5} P_1(x) + 4 P_0(x)$$

## 5.10 A GENERATING FUNCTION OF LEGENDRE'S POLYNOMIAL:

Prove that $Pn(x)$ is the coefficient of $z^n$ in the expansion of $(1 - 2xz + z2)^{-1/2}$ in ascendig powers of $z$.

**Proof.**

$$(1 - 2xz + z2)^{-1/2} = [1 - z(2x - z)]^{-1/2}$$

$$= 1 + \frac{1}{2} z(2x - z) + \frac{-\frac{1}{2}\left(-\frac{3}{2}\right)}{2!} z^2 (2x - z)^2 + \ldots\ldots\ldots.$$

$$+ \frac{-\frac{1}{2}\left(-\frac{3}{2}\right)\frac{-5}{2}\ldots\ldots\left(\frac{-1}{2} - n + 1\right)}{n!} (-z)^n (2x - z)^n + \ldots.. \quad ------(1)$$

Now coefficient of $z^n$ in

$$\frac{-\frac{1}{2}\left(-\frac{3}{2}\right)\frac{-5}{2}\ldots\ldots\left(\frac{-1}{2}-\mathrm{n}+1\right)}{\mathrm{n}!}(-\mathrm{z})^{\mathrm{n}}(2x-z)^{n}=\frac{-\frac{1}{2}\left(-\frac{3}{2}\right)\frac{-5}{2}\ldots\ldots\left(\frac{-1}{2}-n+1\right)}{n!}(-1)^{n}(2x)^{n}$$

$$=\frac{1.3.5\ldots\ldots(2n-1)}{2^{n}\,n!}\ (2)^{n}(x)^{n}$$

$$==\frac{1.3.5\ldots\ldots(2n-1)}{2^{n}\,n!}\ (x)^{n}$$

Coefficient of $z^n$ in

$$\frac{-\frac{1}{2}\left(-\frac{3}{2}\right)\frac{-5}{2}\ldots\ldots\left(\frac{-1}{2}-n+2\right)}{(n-1)!}(-z)^{n-1}(2x-z)^{n-1}$$

$$=\frac{-\frac{1}{2}\left(-\frac{3}{2}\right)\frac{-5}{2}\ldots\ldots\left(\frac{-1}{2}-n+2\right)}{(n-1)!}(-1)^{n}(-(n-1)(2x)^{n-2})$$

$$=\frac{1.3.5\ldots\ldots(2n-3)}{2^{n-1}\,(n-1)!}\ (2)^{n-2}\ (n-1)(x)^{n-1}$$

$$=\frac{1.3.5\ldots\ldots(2n-3)}{2^{1}\,(n-1)!}\ (n-1)(x)^{n-2}$$

$$=\frac{1.3.5\ldots\ldots(2n-3)}{2^{1}\,(n-1)!}\ \frac{(2n-1)}{(2n-1)}(n-1)(x)^{n-2}$$

$$=\frac{1.3.5\ldots\ldots(2n-3)}{n!}\ \frac{(2n-1)}{1}\frac{n(n-1)}{2(2n-1)}(x)^{n-2}$$

Coefficient of $z^n$ in

$$\frac{-\frac{1}{2}\left(-\frac{3}{2}\right)\frac{-5}{2}\ldots\ldots\left(\frac{-1}{2}-n+3\right)}{(n-2)!}(-z)^{n-2}(2x-z)^{n-1}$$

$$=\frac{-\frac{1}{2}\left(-\frac{3}{2}\right)\frac{-5}{2}\ldots\ldots\left(\frac{-1}{2}-n+3\right)}{(n-2)!}(-1)^{n-2}\ \frac{(n-2)(n-3)}{2!}\ (2x)^{n-4}$$

$$=\frac{1.3.5\ldots\ldots(2n-5)}{2^{n-2}\,(n-2)!}\frac{(n-2)(n-3)}{2!}\ (2x)^{n-4}$$

$$= \frac{1.3.5\ldots\ldots(2n-5)(2n-3)(2n-1)}{4\ (n-2)!}\ \frac{(n-2)(n-3)}{2(2n-3)(2n-1)}\ (x)^{n-4}$$

$$= \frac{1.3.5\ldots\ldots(2n-1)}{4n\ (\ n-1\ )\ (n-2)!}\ \frac{n(n-1)(n-2)(n-3)}{2(2n-3)(2n-1)}\ (x)^{n-4}$$

$$= \frac{1.3.5\ldots\ldots((2n-1)}{n!}\ \frac{n(n-1)(n-2)(n-3)}{2.4(2n-3)(2n-1)}\ (x)^{n-4}$$

and so on.

Thus coefficient of *zn* in the expansion of (1)

$$= \frac{1.3.5\ldots\ldots((2n-1)}{n!}\ [\ x^n - \frac{n(n-1)}{2(2n-1)}\ (x)^{n-2}\ + \frac{n(n-1)(n-2)(n-3)}{2.4(2n-3)(2n-1)}\ (x)^{n-4}\ - \ldots\ldots\ldots]$$

$= Pn\ (x)$

Thus coefficients of $z$, $z^2$, $z^3$ ... etc. in (1) are $P_1\ (x)$, $P_2\ (x)$, $P_3\ (x)\ldots$

Hence

$(1 - 2\ xz + z2)^{-1/2} = P_0\ (x) + zP_1\ (x) + z^2P_2(x) + z^3P_3\ (x) + \ldots + z^nPn\ (x) + \ldots$

$$(1 - 2\ xz + z2)^{-1/2} = \sum_{n=0}^{n=\infty} Pn\ (x)\ .\ z^n \qquad \textbf{proved.}$$

**Example:**

Prove that Pn (1) = 1.

**Solution.**

We know that

$(1 - 2\ xz + z2)^{-1/2} = 1 + zP_1\ (x) + z^2P_2(x) + z^3P_3\ (x) + \ldots + z^nPn\ (x) + \ldots$

Substituting 1 for $x$ in the above equation, we get

$(1 - 2\ z + z2)^{-1/2} = 1 + zP_1\ (1) + z^2P_2(1) + z^3P_3\ (1) + \ldots + z^nPn\ (1) + \ldots$

$$((1 - z)^2)^{-1/2} = \sum_{n=0}^{n=\infty} Pn\ (1)\ .\ z^n$$

$$(1 - z))^{-1} = \sum Pn\ (1)\ z^n$$

$$\sum Pn\ (1)\ z^n = (1 - z))^{-1} = 1 + z + z^2 + z^3 + \ldots + z^n + \ldots$$

## 5.11 ORTHOGONALITY OF LEGENDRE POLYNOMIALS:

$$\int_{-1}^{+1} P_m(x).P_n(x)dx = 0 \qquad n \neq m$$

**Proof.**

*Pn* (*x*) is a solution of

$$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + n(n+1)y = 0 \quad \text{------------(1)}$$

*Pm* (*x*) is the solution of

$$(1-x^2)c - 2x\frac{dz}{dx} + m(m+1)z = 0 \quad \text{------------(2)}$$

Multiplying (1) by z and (2) by *y* and subtracting, we get

$$(1-x^2)[z\frac{d^2z}{dx^2} - y\frac{d^2z}{dx^2}] - 2x[z\frac{dz}{dx} - y\frac{dz}{dx}] + (n(n+1) - m(m+1))yz = 0$$

$$(1-x^2)\{[z\frac{d^2z}{dx^2} + \frac{dz}{dx}\frac{dy}{dz}] - [\frac{dy}{dx}\frac{dz}{dx} + y\frac{d^2z}{dx^2}] - 2x[z\frac{dy}{dx} - y\frac{dz}{dx}]$$

$$+ (n-m)(n+m+1)yz = 0$$

$$\frac{d}{dx}[(1-x^2)(z\frac{dz}{dx} - y\frac{dz}{dx})] + (n-m)(n+m+1)yz = 0$$

Now integrating from – 1 to 1, we get

$$[(1-x^2)(\frac{dz}{dx} - y\frac{dz}{dx})]^{-1}{}_{+1} + (n-m)(n+m+1)\int_{-1}^{+1} y.z.dx = 0$$

$$0 + (n-m)(n+m+1)\int_{-1}^{+1} y.z.dx = 0$$

$$\int_{-1}^{+1} P_m(x).P_n(x)dx = 0 \qquad \textbf{Proved.}$$

## 5.12 GREEN'S THEOREM

**Statement:**

*If* ϕ $(x, y)$, ψ $(x, y)$, $\frac{\partial \phi}{\partial y}$ *and* $\frac{\partial \psi}{\partial x}$ be continuous functions over a region R bounded by simple closed cuve C in x-y plane, then

$$\oint_c (\phi\, dx + \psi\, dy) \qquad = \iint_R \left(\frac{\partial \psi}{\partial x} - \frac{\partial \phi}{\partial y}\right) dxdy$$

**Example:**

A vector field $\vec{F}$ is given by $\vec{F}$ = siny $\hat{\imath}$ + x (1+cosy) $\hat{\jmath}$

Evaluate the integral $\int_c \vec{F} \cdot \overrightarrow{dr}$ where C is the circular path given by $x^2 + y^2 = a^2$.

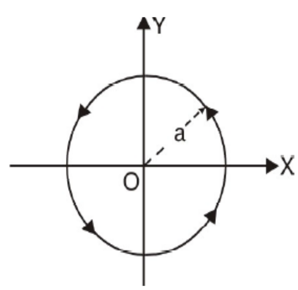
**Solution:**

$$\vec{F} = \text{siny}\ \hat{\imath} + \text{x (1+cosy)}\ \hat{\jmath}$$

$$\int_c \vec{F} \cdot \overrightarrow{dr} = \int_c (\ \text{siny}\ \hat{\imath}\ +\ \text{x}(1 + \text{cosy})\ \hat{\jmath}) \ .\ (\ \hat{\imath}\ \text{dx} +\ \hat{\jmath}\ dy)$$

$$= \text{siny dx + x (1+cosy) dy}$$

On applying green's theorem, we get




$$\oint_c (\phi\, dx + \psi\, dy) \qquad = \iint_s \left(\frac{\partial \psi}{\partial x} - \frac{\partial \phi}{\partial y}\right) dxdy$$

$$= \iint_s ((1 + cosy) - cosy)\text{dxdy}$$

Where s is the circular plane surface of radius a.

$$= \iint_s dxdy$$

$$= \text{area of circle}$$